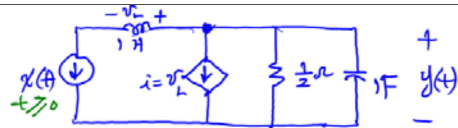
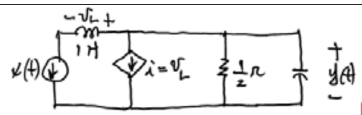
EE103 Lecture 25, Nov. 29, 2017

- HW 9 posted
- Qz 8 on Dec 4, next Monday
  HW 10 will be posted on Dec 4, but no more Qz.
- course Review on Dec. 8



$y(0_-) = 2$

KCL
$$x(t) + x'(t) + 2y(t) + y'(t) = 0$$
$$\Big\downarrow \mathcal{L}$$
$y(0_-) = y(0_+) = 2$
$$X(s) + [sX(s) - x(0_+)] + 2Y(s) + sY(s) - 2 = 0$$
solve for $Y(s)$
$$Y(s)[s+2] = 2 - X(s) - sX(s) + x(0_-)$$
$$= 2 - X(s)[s+1] + x(0_-)$$
$$\left\langle \frac{Y(s)}{X(s)} = -\frac{s+1}{s+2} + \frac{2}{X(s)} \right\rangle \quad Y(s) = \frac{2 + x(0_+)}{s+2} - \frac{s+1}{s+2}X(s)$$



$y(0_-) = 2$

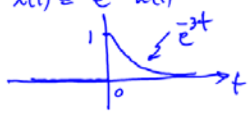
$\boxed{\text{“}x(0_-) = 0\text{”}}$?

$$x + \dot{x} + 2y + \dot{y} = 0$$
where $v_L = \dot{x}$

For $x(t) = e^{-3t}u(t)$, $y(t) = (3e^{-2t} - 2e^{-3t})u(t)$

$y(t)\big|_{t=0} = 2$? $\longleftrightarrow$ 1 incorrect.

$$x(t) = e^{-3t}u(t)$$

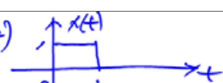


$$Y(s) = \frac{2 + x(0_+)}{s+2} - \frac{s+1}{s+2}X(s)$$

For $x(t) = e^{-3t}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+3}$

$$Y(s) = \frac{2 + x(0)}{s+2} - \left[\frac{s+1}{s+2}\ \frac{1}{s+3}\right]$$
$$= \frac{2 + x(0_+)}{s+2} - \left[\frac{-1}{s+2} + \frac{2}{s+3}\right]$$
$$\Big\downarrow \mathcal{L}^{-1}$$
due to i.c.
$$y(t) = \left((2 + x(0_+))e^{-2t} + e^{-2t} - 2e^{-3t}\right)u(t)$$
$$= \left((x(0_+) + 3)e^{-2t} - 2e^{-3t}\right)u(t)$$
$$y(0) = x(0_+) + 3 - 2 = 2 \Rightarrow \underbrace{x(0_+) = 1}_{\text{required}}$$

(example 2)



$$x(t) = u(t) - u(t-1)$$
$$\Big\downarrow \mathcal{L}$$
$$X(s) = \frac{1}{s} - e^{-1s}\frac{1}{s} = \frac{1}{s}(1 - e^{-s})$$
$$Y(s) = \frac{2 + x(0)}{s+2} - \frac{s+1}{s+2}\ \frac{1}{s}(1 - e^{-s})$$
$$= \frac{2}{s+2} - \frac{s+1}{(s+2)s} - \left[\frac{s+1}{(s+2)s}e^{-s}\right]$$
$-f(t)u(t)$ $\qquad$ $f(t-1)u(t-1)$
$$= \frac{2 + x(0_+)}{s+2} - \left(\frac{\frac{1}{2}}{s+2} + \frac{\frac{1}{2}}{s}\right) - $$
$$y(t) = 3e^{-2t}u(t) - \underbrace{\frac{1}{2}\left(e^{-2t}u(t) + u(t)\right)}_{f(t)} - f(t-1)$$
$y(0) = 2$ ✓

summary
$$x + \frac{dx}{dt} + 2y + \frac{dy}{dt} = 0 \quad \text{(KCL)}$$
or
$$\frac{dy}{dt} + 2y = -\frac{dx}{dt} - x$$
$$\Big\downarrow \mathcal{L}$$
$x(0_+)$
$$sY(s) - y(0_-) + 2Y(s) = -[sX(s) - x(0_-) + X(s)]$$
$$(s+2)Y(s) - y(0_-) = -(s+1)X(s) + x(0_-)$$
Neglecting initial conditions ($x(0_-) = 0$, $y(0_-) = 0$)

particular solution
$$\frac{Y(s)}{X(s)} = -\left(\frac{s+1}{s+2}\right) = -1 + \left[\frac{1}{s+2}\right] \text{ proper}$$
In General,
$$H(s) = \frac{Y(s)}{X(s)} = H_1(s) + \left[\frac{b_m s^m + b_{m-1}s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}\right]$$
where $m < n$ (proper form)

In the previous example $H_1(s) = -1$

$$H(s) - H_1(s) = \frac{1}{s+2} = \frac{b_0}{a_1 s + a_0}$$

$n=1$, $m=0$ $\qquad \frac{1}{s-(-2)}$, $-2 = p_1$

$\boxed{\text{Let us consider}}$

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{\cancel{a_n} s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

proper system $\qquad a_n = 1$

$$= \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \quad (m<n)$$

$$= \frac{b_m (s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

p_k, $k=1,n$ is the kth pole at which $D(s)\big|_{s=p_k}=0$

z_k, $k=1,m$ is the kth zero at which $N(s)\big|_{s=z_k}=0$

For $H(s) = \frac{1}{s+2} = \frac{1}{s-(-2)} = \frac{1}{s-p}$ $\qquad \boxed{p=-2}$

(Case 1) all poles are distinct

$$H(s) = \frac{N(s)}{D(s)} = \sum_{k=1}^{n} \frac{K_k}{s-p_k}$$

$$\boxed{K_k = H(s)(s-p_k)\big|_{s=p_k}}$$

Proof $\quad H(s) = \frac{K_1}{s-p_1} + \frac{K_2}{s-p_2} + \cdots + \frac{K_k}{s-p_k} + \cdots + \frac{K_n}{s-p_n}$

$$H(s)(s-p_k)\Big|_{s=p_k} = \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{K_j (s-p_k)}{s-p_j}\Bigg|_{s=p_k} + K_k$$

$$\underset{s=p_k}{=} \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{0}{s-p_j} + K_k$$

Example $\quad H(s) = \frac{s+4}{s^2+3s+2} = \frac{s+4}{(s+1)(s+2)}$

$$= \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

$K_1 = H(s)(s+1)\big|_{s=-1} = \frac{-1+4}{-1+2} = 3$ ✓

$K_2 = H(s)(s+2)\big|_{s=-2} = \frac{-2+4}{-2+1} = -2$

$$H(s) = \frac{3}{s-\underset{p_1}{(-1)}} + \frac{-2}{s-\underset{p_2}{(-2)}}$$

$\big\downarrow \mathcal{L}^{-1}$

$$h(t) = \left(3e^{-t} - 2e^{-2t}\right)u(t)$$

Case 2. with poles repeated (double, triple, etc.)

$$H(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s-p_i)^K D_1(s)}$$

$$= \frac{K_{i1}}{s-p_i} + \frac{K_{i2}}{(s-p_i)^2} + \cdots + \frac{K_{iK-1}}{(s-p_i)^{K-1}} + \frac{K_{iK}}{(s-p_i)^K} + H_1(s)$$

$$h(t) = K_{i1} e^{p_i t} + K_{i2}\left(t e^{p_i t}\right) + \cdots K_{iK}\left(t^{K-1} e^{p_i t}\right) + h_1(t)$$

✓ How to find K_{ij}, $j=1,K$?

Recall $\quad H(s) = \frac{K_{i1}}{s-p_i} + \frac{K_{i2}}{(s-p_i)^2} + \cdots + \frac{K_{iK-1}}{(s-p_i)^{K-1}} + \frac{K_{iK}}{(s-p_i)^K}$

First find K_{iK} by multiplying both sides by $(s-p_i)^K$

$$H(s)(s-p_i)^K = \sum_{j=1}^{K-1} \frac{K_{ij}}{(s-p_i)^j}(s-p_i)^K + K_{iK}$$

set $s=p_i$, then

$$\boxed{H(s)(s-p_i)^K\big|_{s=p_i} = K_{iK}}$$

Next find K_{iK-1}

$$\boxed{\frac{d}{ds} H(s)(s-p_i)^K \Big|_{s=p_i} = K_{iK-1}}$$

$$K_{iK-2} = \frac{1}{2!}\frac{d^2}{ds^2} H(s)(s-p_i)^K \Big|_{s=p_i}$$

$\vdots$

$$K_{i1} = \frac{1}{(K-1)!}\frac{d^{K-1}}{ds^{K-1}} H(s)(s-p_i)^K \Big|_{s=p_i}$$

Example

$$H(s) = \frac{1}{(s+1)^3(s+2)}$$

$$H(s) = \frac{K_{11}}{s+1} + \frac{K_{12}}{(s+1)^2} + \frac{K_{13}}{(s+1)^3} + \frac{K_2}{s+2}$$

$K_2 = H(s)(s+2)\big|_{s=-2} = -1$

$K_{13} = H(s)(s+1)^3\big|_{s=-1} = \frac{1}{-1+2} = 1$

$$K_{12} = \frac{d}{ds} H(s)(s+1)^3 \Big|_{s=-1} = \frac{d}{ds}\left(\frac{1}{s+2}\right)\Big|_{s=-1} = \frac{-1}{(s+2)^2}\Big|_{s=-1} = -1$$

$$K_{11} = \frac{1}{2!}\frac{d^2}{ds^2}\left(H(s)(s+1)^3\right)\Big|_{s=-1} = \frac{1}{2!}\frac{d^2}{ds^2}\frac{1}{(s+2)}\Big|_{s=-1} = \frac{1}{2!}\frac{2(s+2)}{(s+2)^4}\Big|_{s=-1} = \frac{1}{2!}\frac{2}{(s+2)^3}\Big|_{s=-1} = 1$$

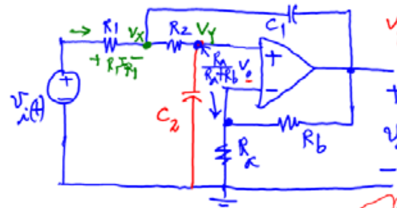
thus,

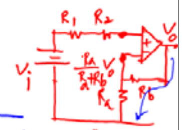
$$H(s) = \frac{1}{s+1} - \frac{1}{(s+1)^2} + \frac{1}{(s+1)^3} - \frac{1}{s+2}$$

$$\downarrow \mathcal{L}^{-1}$$

$$h(t) = \left(e^{-t} - t e^{-t} - \frac{1}{2} t^2 e^{-t} - e^{-2t}\right) u(t)$$

Sallen-Key Filter example





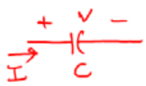
$$V_i = \frac{R_a}{R_a+R_b} V_o \qquad \frac{V_o}{V_i} = 1 + \frac{R_b}{R_a}$$

$$= C_1 s (V_x - V_o) + C_2 s V_Y$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = ? \qquad I_{R_1} = \frac{V_i - V_x}{R_1}$$

$$V_i(s) = R_1 I_{R_1} + V_x$$

$$V_x = C_2 s \overbrace{\left(\frac{R_a}{R_a+R_b}\right) V_o}^{V_Y} R_2 + V_Y$$



$$I(s) = \underbrace{[Cs]}_{\text{admittance}} V(s) = \frac{V(s)}{\underbrace{\left(\frac{1}{Cs}\right)}_{=\text{Impedance}}}$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \left[\frac{\omega_o^2}{s^2 + 2\alpha s + \omega_o^2}\right] = \frac{K_1}{s-p_1} + \frac{K_2}{s-p_2}$$

$$\omega_o^2 = \frac{1}{R_1 R_2 C_1 C_2} \qquad \omega_o = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}$$