EE103 lect #18 Nov. 8, 2017

HW 6 posted

QZ 6 on Nov. 13

Parseval's Theorem

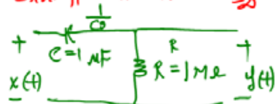
$$\boxed{\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega}$$

$$f(t) = \frac{1}{2\pi}\int F(\omega) e^{+j\omega t} d\omega$$

Proof $\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} f(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{+j\omega t} d\omega\right) dt$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) \underbrace{\int_{-\infty}^{\infty} f(t) e^{+j\omega t} dt}_{F(-\omega) = F^*(\omega)} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \quad \boxtimes$$

Example to show $\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$



$$H(s) = \frac{R}{R + \frac{1}{Cs}} = \frac{RCs}{1 + RCs} = \frac{s}{s+1}$$

when input $u(t)$ $\mathcal{L}[u(t)] = \frac{1}{s}$

$$H(s) = \frac{s}{s+1}$$

$$Y(s) = X(s)\,H(s) = \frac{1}{s}\cdot\frac{s}{s+1} = \frac{1}{s+1}$$

$$\underline{y(t) = e^{-t}u(t)} \xleftrightarrow{\mathcal{F}} \frac{1}{1+j\omega}$$

$$\int_{-\infty}^{\infty} |y(t)|^2 dt = \int_0^{\infty} e^{-2t} dt = \left.\frac{e^{-2t}}{-2}\right|_0^{\infty} = +\frac{1}{2}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{1}{1+j\omega}\right|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+\omega^2} d\omega$$

$$= \frac{1}{2\pi}\, 2\int_0^{\infty} \frac{1}{1+\omega^2} d\omega \quad = \tan^{-1}\infty = \frac{\pi}{2}$$

$$= \frac{1}{\cancel{2\pi}}\,\cancel{2}\,\frac{\cancel{\pi}}{\cancel{2}} = \frac{1}{2}$$

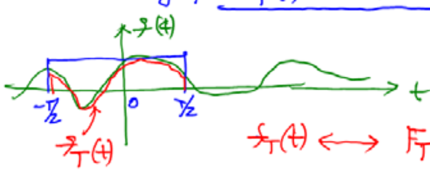
For power signal $f(t)$,

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2 dt < \infty$$

(examples) $u(t)$, $sgn(t)$, periodic signals

But their energy is infinite!

Truncated signal $f_T(t) = f(t)\, rect(t/T)$



$$f_T(t) \longleftrightarrow F_T(\omega)$$

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} |f_T(t)|^2 dt$$

$$\int_{-\infty}^{\infty} |f_T(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F_T(\omega)|^2 d\omega$$

Parseval's Theorem

$$P = \lim_{T\to\infty} \frac{1}{T}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |F_T(\omega)|\, d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\lim_{T\to\infty}\frac{1}{T}|F_T(\omega)|^2}_{\mathcal{P}_f(\omega)\ \text{power spectral density}} d\omega$$

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{P}_f(\omega)\, d\omega$$

from the Fourier series of $f(t)$

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t}$$

$$F(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} F(k\omega_0)\,\delta(\omega - k\omega_0)$$

Power $P = \sum_{k=-\infty}^{\infty} |c_k|^2 = c_0^2 + 2\sum_{k=1}^{\infty}|c_k|^2$

$$= \frac{1}{4\pi^2}F(0) + \frac{1}{2\pi^2}\sum_{k=1}^{\infty}|F(k\omega_0)|^2$$

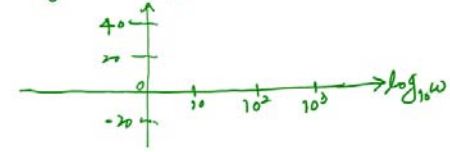
$$\frac{1}{2\pi}F(k\omega_0) = c_k$$

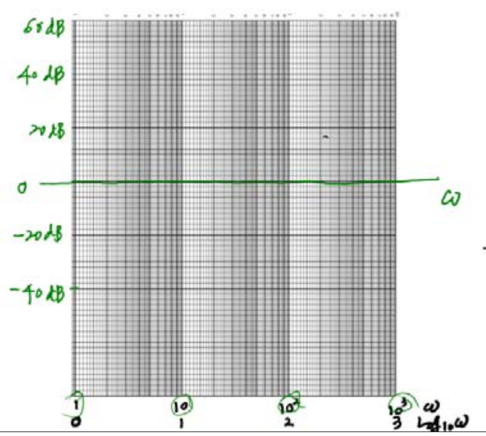
$$Y(\omega) = H(\omega)X(\omega)$$

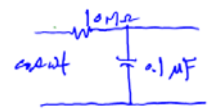
$$|Y(\omega)|^2 = |H(\omega)|^2|X(\omega)|^2$$

$$10\log_{10}|Y(\omega)|^2 = \underbrace{20\log_{10}|H(\omega)|} + 20\log_{10}|X(\omega)|$$

$20\log_{10}|H(\omega)|$ [dB]





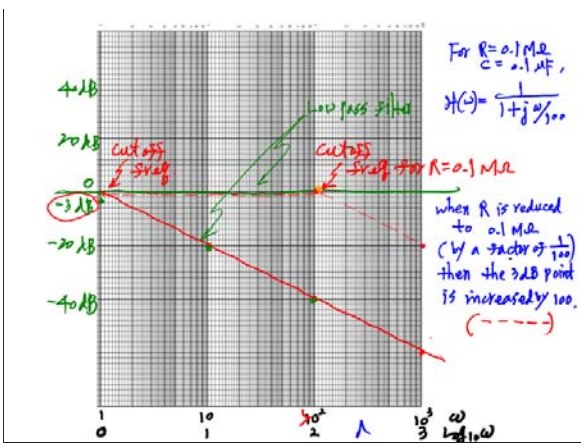


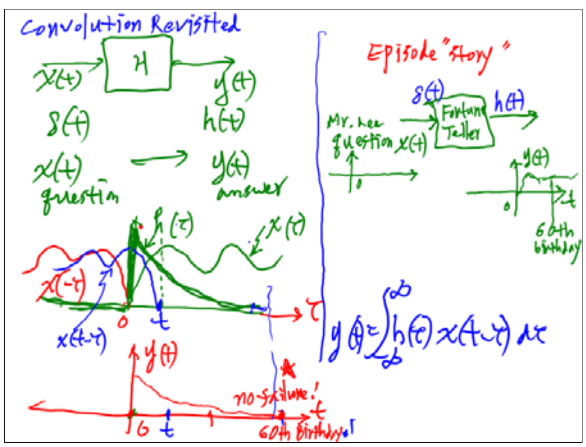
$$H(\omega) = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} = \frac{1}{1 + j\omega RC}$$

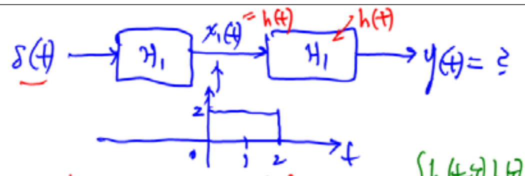
For $R = 10^7$, $C = 10^{-7}$: $H(\omega) = \dfrac{1}{1 + j\omega(10^7 \cdot 10^{-7})} = \dfrac{1}{1 + j\omega}$

$$|H(\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$

- $\omega = 1$: $\frac{1}{\sqrt{2}}$, $20\log 2^{-\frac{1}{2}} = -10\log_{10} 2 = -3$ dB
- 10: $\frac{1}{10}$, $20\log_{10} 10^{-1} = -20$ dB
- 10^2: $\frac{1}{10^2}$, $20\log_{10} 10^{-2} = -40$ dB

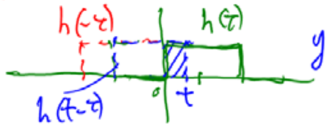






$y(t) = h(t) * h(t)$

$\int h(t-\tau)h(\tau)d\tau$

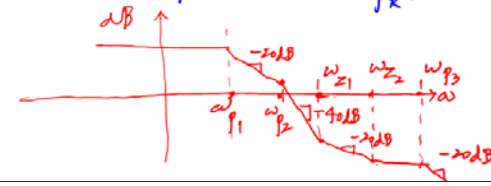


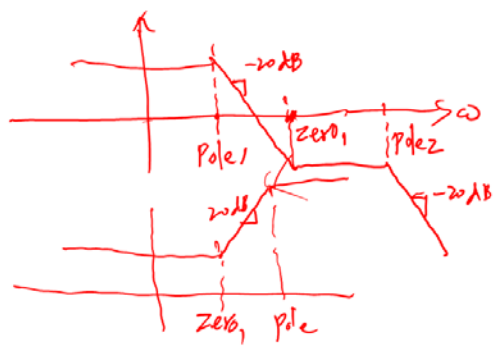
$y(t) = 4t$

In general

$$H(\omega) = K\frac{(1+j\frac{\omega}{\omega_{z_1}})(1+j\frac{\omega}{\omega_{z_2}})\cdots(1+j\frac{\omega}{\omega_{z_M}})}{(1+j\frac{\omega}{\omega_{p_1}})(1+j\frac{\omega}{\omega_{p_2}})\cdots(1+j\frac{\omega}{\omega_{p_N}})}$$

$$20\log_{10}|H(\omega)| = 20\log_{10}|K| + \sum_{j=1}^{M} 20\log_{10}\left|1+j\frac{\omega}{\omega_{z_j}}\right| - \sum_{k=1}^{N} 20\log_{10}\left|1+j\frac{\omega}{\omega_{p_k}}\right|$$





$$H(s) = K\frac{(s+z_1)(s+z_2)\cdots}{(s+p_1)(s+p_2)\cdots}$$

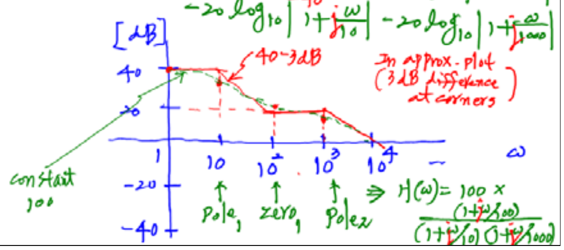
$$\overset{s=j\omega}{=} K\frac{(j\omega+z_1)(j\omega+z_2)\cdots}{(j\omega+p_1)(j\omega+p_2)\cdots}$$

$$= K\frac{p_1p_2\cdots(1+j\omega/z_1)(1+j\omega/z_2)\cdots}{z_1z_2\cdots(1+j\omega/p_1)(1+j\omega/p_2)\cdots}$$

Example

$$H(\omega) = 100\frac{(1+j\frac{\omega}{100})}{(1+j\frac{\omega}{10})(1+j\frac{\omega}{1000})}$$

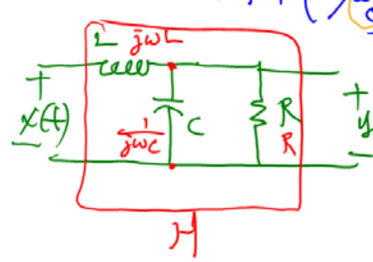
$$20\log_{10}|H(\omega)| = \underbrace{20\log_{10}100}_{=40} + 20\log_{10}\left|1+j\frac{\omega}{100}\right| - 20\log_{10}\left|1+j\frac{\omega}{10}\right| - 20\log_{10}\left|1+j\frac{\omega}{1000}\right|$$



$$\Rightarrow H(\omega) = 100\times\frac{(1+j\omega/100)}{(1+j\omega/10)(1+j\omega/1000)}$$

N-th order Butterworth Filter

$$|H(\omega)| = \frac{1}{\sqrt{1+(\omega/\omega_c)^{2N}}}$$



$$H(\omega) = \frac{\frac{1}{j\omega C} \parallel R}{j\omega L + \frac{1}{j\omega C} \parallel R}$$

$$= \frac{1}{j\omega L\left(j\omega C + \frac{1}{R}\right) + 1}$$

$$= \frac{1}{(1 - \omega^2 LC) + j\omega \frac{L}{R}}$$

$$= \frac{1}{\left(1 - \left(\frac{\omega}{\omega_c}\right)^2\right) + j\omega \frac{\omega_c^{2}}{RC}}$$

$$\omega_c = \frac{1}{\sqrt{LC}}$$

For $L = 2R^2C$

$$H(\omega) = \frac{\omega_c^2}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^4}}$$

Butterworth filter $N = 2$