EE103 Lecture 12 Oct 25, 2017

Midterm Nov. 1, 2017 (W)

on the EE103 Fall 2017 webpage

✓ Midterm Exam of Sp. 2017 (prep 1)

✓ Exercise problems (prep 2)

Nov. 1 Midterm Exam $\subset$ {prep1, prep2, Quiz 4}

1 page – Tables + formulas allowed.
$8\frac{1}{2} \times 11$"

Fourier Series

TABLE 4.2 Forms of the Fourier Series

Name	Equation
Exponential	$\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$; $C_k = \lvert C_k \rvert e^{j\theta_k}, C_{-k} = C_k^*$
Combined trigonometric	$C_0 + \sum_{k=1}^{\infty} 2\lvert C_k \rvert \cos(k\omega_0 t + \theta_k)$
Trigonometric	$A_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$ $2C_k = A_k - jB_k, C_0 = A_0$
Coefficients	$C_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$

For a periodic signal $x(t)$ with its minimum period T_0, $x(t + T_0) = x(t)$ for all t.

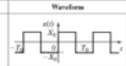
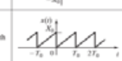
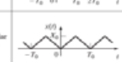
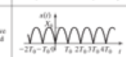
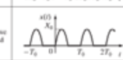
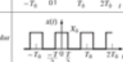
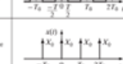
$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \quad \left(\omega_0 = \frac{2\pi}{T_0}\right)$$

$$\boxed{C_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt}$$

Alternatively

$$C_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

TABLE 4.3 Fourier Series for Common Signals

Name	Waveform	C_0	$C_k, k \neq 0$	Comments
1. Square wave		0	$-j\frac{2X_0}{\pi k}$	$C_k = 0$, k even
2. Sawtooth		$\frac{X_0}{2}$	$j\frac{X_0}{2\pi k}$	
3. Triangular wave		$\frac{X_0}{2}$	$\frac{-2X_0}{(\pi k)^2}$	$C_k = 0$, k even
4. Full-wave rectified		$\frac{2X_0}{\pi}$	$\frac{-2X_0}{\pi(4k^2-1)}$	
5. Half-wave rectified		$\frac{X_0}{\pi}$	$\frac{-X_0}{\pi(k^2-1)}$	$C_k = 0$, k odd, except $C_1 = -j\frac{X_0}{4}$ and $C_{-1} = j\frac{X_0}{4}$
6. Rectangular wave		$\frac{TX_0}{T_0}$	$\frac{TX_0}{T_0} \operatorname{sinc} \frac{Tk\omega_0}{2}$	$\frac{Tk\omega_0}{2} = \frac{\pi Tk}{T_0}$
7. Impulse train		$\frac{X_0}{T_0}$	$\frac{X_0}{T_0}$	

see EX 4.2 on page 165

See EX 4.6 on page 171

covered in Lect #9

In general, for $x(t)$ periodic with period T_0 (fundamental freq ω_0) $y(t) = H\,x(t)$ can be represented in frequency domain as

$$\sum_{k=-\infty}^{\infty} C_{kx} e^{+jk\omega_0 t} \longrightarrow \boxed{H(jk\omega_0)} \longrightarrow \sum_{k=-\infty}^{\infty} C_{ky} e^{+jk\omega_0 t}$$

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

$$Y(j\omega) = H(j\omega) X(j\omega)$$

or $Y(\omega) = H(\omega) X(\omega) \leftrightarrow Y(s) = H(s) X(s)$

Fourier

Laplace (chap. 7)

$$C_{kx} = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{2\pi} \frac{2\pi}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

$$\lim_{T_0 \to \infty} C_{kx} = \frac{1}{2\pi} \lim_{T_0 \to \infty} \frac{2\pi}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{2\pi} d\omega \underbrace{\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}_{\text{Fourier Transform } \mathcal{F}}$$

$\lim_{T_0 \to \infty} \frac{2\pi}{T_0} = \omega_0 = d\omega$

$k\omega_0 = \omega$

$$= \frac{1}{2\pi} \left[X(\omega) \right] d\omega$$

$$x(t) = \sum_{k=-\infty}^{\infty} C_{kx}\, e^{+jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty}\left[\frac{1}{2\pi}X(\omega)\,d\omega\right]e^{jk\omega_0 t}$$

$$\lim_{T_0\to\infty} x(t) = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{j\omega t}\,d\omega}_{\text{Inverse Fourier Transform}}$$

$$\boxed{\begin{aligned}\mathcal{F}[x(t)] &= \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt = X(\omega)\\ \mathcal{F}^{-1}[X(\omega)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{+j\omega t}\,d\omega = x(t)\end{aligned}}$$

Consider $x(t) = X_0\,\mathrm{rect}(t|T)$

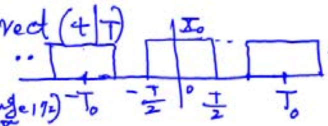


Table 4.3 case 6 (Page 172) $C_0 = \frac{TX_0}{T_0}$

$k \neq 0$ $\quad C_k = \frac{TX_0}{T_0}\,\mathrm{sinc}\left(\frac{Tk\omega_0}{2}\right)$

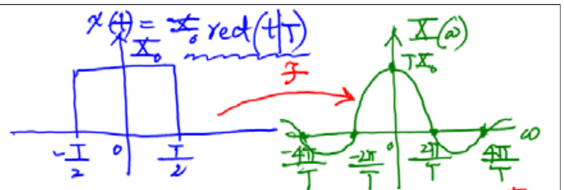
$\left(\mathrm{sinc}\,x = \frac{\sin x}{x}\right)$

$$x(t) = C_0 + \sum_{k=1}^{\infty} 2|C_k|\cos(k\omega_0 t + \theta_k) \quad (\text{Ref. Table 4.2})$$

$$2|C_k| = \frac{2TX_0}{T_0}\,\mathrm{sinc}\,\frac{Tk\omega_0}{2}$$

$$\theta_k = \begin{cases} 0^\circ, & \mathrm{sinc}(Tk\omega_0/2) > 0\\ 180^\circ, & \mathrm{sinc}(Tk\omega_0/2) < 0\end{cases}$$

$x(t) = X_0\,\mathrm{rect}(t|T)$



$X(\omega) = X_0 T\,\mathrm{sinc}\,\frac{\omega T}{2}$

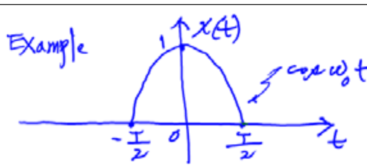
$$X(\omega) = \int x(t)\,e^{-j\omega t}\,dt$$

$$= \int_{-T/2}^{T/2} X_0\, e^{-j\omega t}\,dt$$

$$= X_0\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T/2}^{T/2} = 2X_0\,\frac{e^{j\frac{\omega T}{2}} - e^{-j\frac{\omega T}{2}}}{2j\omega} = 2X_0\,\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\,\frac{T}{2}$$

$$= \boxed{X_0 T\,\mathrm{sinc}(\omega T/2)}$$

Example



$x(t) = 0$ at $t = \frac{T}{2}, -\frac{T}{2}$

$\Rightarrow \cos\omega_0\left(\frac{T}{2}\right) = \cos\omega_0\left(-\frac{T}{2}\right) = 0$

thus $\frac{\omega_0 T}{2} = \frac{\pi}{2}$

$$X(\omega) = \int x(t)\,e^{-j\omega t}\,dt = \int_{-T/2}^{T/2}\cos\omega_0 t\cdot e^{-j\omega t}\,dt$$

$$= \int_{-T/2}^{T/2}\left(\frac{e^{+j\omega_0 t} + e^{-j\omega_0 t}}{2}\right)e^{-j\omega t}\,dt$$

$$= \int_{-T/2}^{T/2}\frac{e^{-j(\omega-\omega_0)t} + e^{-j(\omega+\omega_0)t}}{2}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-j(\omega-\omega_0)t}}{-j(\omega-\omega_0)} + \frac{e^{-j(\omega+\omega_0)t}}{-j(\omega+\omega_0)}\right]_{-T/2}^{T/2}$$

$$= \frac{1}{2}\left[\frac{e^{-j(\omega-\omega_0)\frac{T}{2}} - e^{-j(\omega-\omega_0)\left(-\frac{T}{2}\right)}}{-j(\omega-\omega_0)} + \frac{e^{-j(\omega+\omega_0)\frac{T}{2}} - e^{-j(\omega+\omega_0)\left(-\frac{T}{2}\right)}}{-j(\omega+\omega_0)}\right]$$

$$= \frac{1}{2}\left[\frac{e^{-j\frac{\omega T}{2}}(+j) - e^{+j\omega T/2}(-j)}{-j(\omega-\omega_0)} + \frac{e^{-j\frac{\omega T}{2}}(-j) - e^{+j\frac{\omega T}{2}}(+j)}{-j(\omega+\omega_0)}\right]$$

$e^{\pm j\omega_0 T/2} = \pm j$

$$= \frac{1}{2}\left[\frac{+j\left(e^{-j\frac{\omega T}{2}} + e^{+j\frac{\omega T}{2}}\right)}{-j(\omega-\omega_0)} + \frac{-j\left(e^{-j\frac{\omega T}{2}} + e^{+j\frac{\omega T}{2}}\right)}{-j(\omega+\omega_0)}\right]$$

$$= \frac{1}{2}\left[\frac{-2\cos\frac{\omega T}{2}}{\omega-\omega_0} + \frac{2\cos\frac{\omega T}{2}}{\omega+\omega_0}\right]$$

$$= \cos\frac{\omega T}{2}\left[\frac{-1}{\omega-\omega_0} + \frac{1}{\omega+\omega_0}\right]$$

$$= \cos\frac{\omega T}{2}\left[\frac{-(\omega+\omega_0)+\omega-\omega_0}{(\omega-\omega_0)(\omega+\omega_0)}\right]$$

$$= \frac{-2\omega_0\cos\frac{\omega T}{2}}{\omega^2-\omega_0^2} = \frac{2\omega_0\cos\frac{\omega T}{2}}{\omega_0^2-\omega^2} = X(\omega)$$

At $\omega = 0$, $X(0) = \frac{2\omega_0\cos 0}{\omega_0^2} = \frac{2}{\omega_0}$

$\omega = \frac{\pi}{T}$, $X\left(\frac{\pi}{T}\right) = \frac{2\omega_0\cos\left(\frac{\pi}{T}\frac{T}{2}\right)}{\omega_0^2 - \left(\frac{\pi}{T}\right)^2} = 0$

$\omega = \frac{2\pi}{T}$, $X\left(\frac{2\pi}{T}\right) = \frac{2\omega_0\cos\left(\frac{2\pi}{T}\frac{T}{2}\right)}{\omega_0^2 - \left(\frac{2\pi}{T}\right)^2} = \frac{-2\omega_0}{\omega_0^2 - (2\pi/T)^2}$

Sufficient condition for Existence of FT of $f(t)$

Dirichlet conditions

1. a) $f(t)$ is bounded
   b) $f(t)$ has a finite number of maxima, minima
   c) " " discontinuities
2. $f(t)$ is absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |f(t)|\,dt < \infty$$

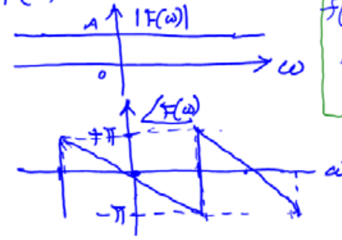
Example $f(t) = A\,\delta(t-t_0)$

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} A\,\delta(t-t_0)\,e^{-j\omega t}\,dt$$

$$= A\int_{-\infty}^{\infty} e^{-j\omega t_0}\,\delta(t-t_0)\,dt = \underline{A e^{-j\omega t_0}}$$

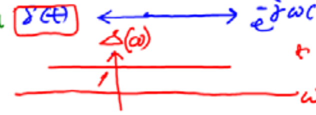
$$F(\omega) = A e^{-j\omega t_0}$$

$$\boxed{\begin{array}{c} f(t) = A\,\delta(t-t_0) \\ \updownarrow \\ F(\omega) = A e^{-j\omega t_0} \end{array}}$$



Let $A = 1$, $t_0 = 0$

then $\boxed{\delta(t)} \longleftrightarrow e^{-j\omega(0)} = \boxed{1}$



← phase angle $= 0\ \forall\,\omega$

For $F(\omega) = \delta(\omega-\omega_0)$

$$\mathcal{F}^{-1}F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)\,e^{+j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\,e^{j\omega_0 t}\,\underbrace{\int_{-\infty}^{\infty}\delta(\omega-\omega_0)\,d\omega}_{=1}$$

$$= \frac{1}{2\pi}\,e^{j\omega_0 t}$$

thus $F(\omega) = \delta(\omega-\omega_0) \longleftrightarrow \frac{1}{2\pi}e^{j\omega_0 t} = f(t)$

$$\boxed{e^{j\omega_0 t} \underset{\mathcal{F}}{\longleftrightarrow} 2\pi\,\delta(\omega-\omega_0)}$$

$u(t) \longrightarrow\ ?$

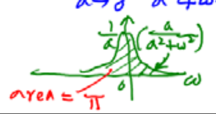
$$U(\omega) = \int_{-\infty}^{\infty} u(t)\,e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-j\omega t}\,dt$$

$$= \frac{-1}{j\omega}\,e^{-j\omega t}\Big|_0^{\infty} = \frac{1}{j\omega} - \underbrace{\frac{1}{j\omega}e^{-j\omega\infty}}_{?}$$

If we take $u(t) = \lim_{a\to 0} e^{-at}\,u(t)$?

$$U(\omega) = \lim_{a\to 0}\int_{-\infty}^{\infty} e^{-at}\,u(t)\,e^{-j\omega t}\,dt = \lim_{a\to 0}\frac{1}{a+j\omega}$$

$$= \lim_{a\to 0}\frac{a-j\omega}{a^2+\omega^2} = \underbrace{\lim_{a\to 0}\frac{a}{a^2+\omega^2}}_{=\pi\delta(\omega)} + \frac{1}{j\omega}$$



$$\int_{-\infty}^{\infty}\frac{a}{a^2+\omega^2}\,d\omega = \tan^{-1}\frac{\omega}{a}\Big|_{-\infty}^{\infty} = \pi$$

as $a \to 0$, ω squizzed the value becomes $\pi\,\delta(\omega)$.

TABLE 5.1 Fourier Transform Properties

Operation	Time Function	Fourier Transform
Linearity	$af_1(t) + bf_2(t)$	$aF_1(\omega) + bF_2(\omega)$
Time shift	$f(t-t_0)$	$F(\omega)e^{-j\omega t_0}$
Time scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$
Time transformation	$f(at-t_0)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)e^{-j\omega t_0/a}$
Duality	$F(t)$	$2\pi f(-\omega)$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
Convolution	$f_1(t)*f_2(t)$	$F_1(\omega)F_2(\omega)$
	$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$
Differentiation	$\frac{d^n[f(t)]}{dt^n}$	$(j\omega)^nF(\omega)$
	$(-jt)^nf(t)$	$\frac{d^n[F(\omega)]}{d\omega^n}$
Integration	$\int_{-\infty}^{t}f(\tau)d\tau$	$\frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$

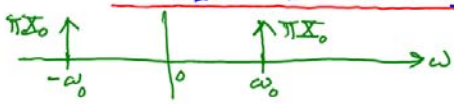
$$f(t) = X_0 \cos \omega_0 t$$

$$\mathcal{F}[f(t)] = \mathcal{F}\left[X_0 \frac{e^{+j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]$$

$$= \frac{X_0}{2}\left(\mathcal{F}\left[e^{+j\omega_0 t}\right] + \mathcal{F}\left[e^{-j\omega_0 t}\right]\right)$$

$$= \frac{X_0}{2}\left[2\pi\,\delta(\omega-\omega_0) + 2\pi\,\delta(\omega+\omega_0)\right]$$

$$= \pi X_0\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

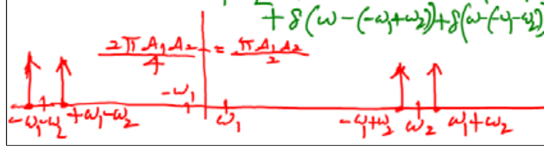


$$f_2(t)\quad f(t) = (A_1 \cos \omega_1 t)(A_2 \cos \omega_2 t)$$

$$f_1(t) \rightarrow \otimes \rightarrow f(t) = A_1 A_2 \frac{e^{+j\omega_1 t} + e^{-j\omega_1 t}}{2}\;\frac{e^{+j\omega_2 t} + e^{-j\omega_2 t}}{2}$$

$$= \frac{A_1 A_2}{4}\left[e^{+j(\omega_1+\omega_2)t} + e^{+j(\omega_1-\omega_2)t} + e^{+j(-\omega_1+\omega_2)t} + e^{+j(-\omega_1-\omega_2)t}\right]$$

$$\mathcal{F}[f(t)] = 2\pi\frac{A_1A_2}{4}\left[\delta(\omega-(\omega_1+\omega_2)) + \delta(\omega-(\omega_1-\omega_2)) + \delta(\omega-(-\omega_1+\omega_2)) + \delta(\omega-(-\omega_1-\omega_2))\right]$$



$$x(t)\,e^{+j\omega_0 t} \xrightarrow{\mathcal{F}} ?$$

$$\int x(t)\,e^{+j\omega_0 t}\,e^{-j\omega t}\,dt$$

$$= \int x(t)\,e^{-j(\omega-\omega_0)t}\,dt$$

$$x(t) \leftrightarrow X(\omega) \qquad = X(\omega-\omega_0)$$

$$\boxed{\begin{array}{l} x(t) \longleftrightarrow X(\omega) \\ x(t)\,e^{+j\omega_0 t} \longleftrightarrow X(\omega-\omega_0) \end{array}}$$

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$$

$$\delta(t) \longrightarrow h(t)$$

$$y(t) = h(t) * x(t)$$

$$Y(\omega) = \int y(t)\,e^{-j\omega t}\,dt$$

$$= \int\left(\int h(t-\tau)\,x(\tau)\,d\tau\right) e^{-j\omega t}\,dt$$

$$\left(\begin{array}{l} t-\tau \rightarrow t' \\ \text{then } t = t'+\tau \\ dt = dt' \end{array}\right) \quad \int\int h(t')\,x(\tau)\,d\tau\,e^{-j\omega(t'+\tau)}\,dt'$$

$$= \int h(t')\,e^{-j\omega t'}\,dt' \int x(\tau)\,e^{-j\omega\tau}\,d\tau$$

$$= H(\omega)\,X(\omega)$$

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t) = h(t) * x(t)$$

$$X(\omega) \qquad H(\omega) \qquad Y(\omega)$$

$$\boxed{Y(\omega) = H(\omega)\,X(\omega)} \quad (\text{freq. domain})$$

Recall $Y(s) = H(s)\,X(s)$ (s domain, $s = \alpha + j\omega$)

$$f(t) = f_1(t)\,f_2(t)$$

$$\mathcal{F}[f(t)] = ?$$

consider $\mathcal{F}^{-1}\left[F_1(\omega) * F_2(\omega)\right]$

$$= \frac{1}{2\pi}\int \int F_1(\omega - z)\,F_2(z)\,dz\;e^{-j\omega t}\,d\omega$$

$$\begin{array}{l} \omega - z = \omega' \\ \omega = \omega' + z \end{array} \quad \frac{1}{2\pi}\int\int F_1(\omega')\,F_2(z)\,dz\;e^{-j(\omega'+z)t}\,d\omega'$$

$$= 2\pi\left(\frac{1}{2\pi}\int F_1(\omega')\,e^{-j\omega' t}\,d\omega'\right)\left(\frac{1}{2\pi}\int F_2(z)\,e^{-jzt}\,dz\right)$$

$$= 2\pi\,f_1(t)\,f_2(t) \qquad \boxed{f_1(t)\,f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} F_1(\omega) * F_2(\omega)}$$

$$\frac{1}{2\pi}\int \delta(\omega-\omega_0)\, e^{+j\omega t}\, d\omega = \frac{1}{2\pi} e^{+j\omega_0 t}\int \delta(\omega-\omega_0)\, d\omega$$

$$\boxed{e^{+j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)}$$

when $\omega_0 = 0$

$$1 \longleftrightarrow 2\pi\,\delta(\omega)$$

$$f(t) \longleftrightarrow F(\omega)$$

$$\boxed{\frac{df(t)}{dt}} \longleftrightarrow ?$$

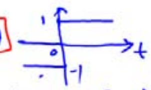
$$\int \frac{d}{dt} f(t)\, e^{-j\omega t}\, dt = f(t)\, e^{-j\omega t}$$

$$-\int f(t)(-j\omega)\, e^{-j\omega t}\, dt \Big|_{-\infty}^{\infty} = \boxed{+j\omega F(\omega)}$$

$$\boxed{\int_{-\infty}^{t} f(t)\, dt} \longleftrightarrow ?$$

$$\int_{-\infty}^{t} f(t)\, dt = \int_{-\infty}^{\infty} f(\tau)\, u(t-\tau)\, d\tau = f(t) * u(t)$$

$\boxed{sgn(t)}$



$$\longleftrightarrow \int_{-\infty}^{0} (-1)\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-j\omega t}\, dt$$

$$= \frac{-e^{-j\omega t}}{j\omega}\Big|_{-\infty}^{0} + \frac{e^{-j\omega t}}{-j\omega}\Big|_{0}^{\infty}$$

$$= \frac{1}{j\omega} + \frac{1}{j\omega} = \boxed{\frac{2}{j\omega}}$$

$$u(t) = \frac{1}{2}(1 + sgn(t)) \longleftrightarrow \pi\,\delta(\omega) + \frac{1}{j\omega}$$

thus $\boxed{\int_{-\infty}^{t} f(t)\, dt} \longleftrightarrow \mathcal{F}[f(t) * u(t)] = F(\omega)\, U(\omega)$

$$= F(\omega)\left[\pi\,\delta(\omega) + \frac{1}{j\omega}\right]$$

$$= \boxed{F(0)\,\pi\,\delta(\omega) + \frac{F(\omega)}{j\omega}}$$

$$\mathcal{F}[f(at)] = ?$$

$$\int f(at)\, e^{-j\omega t}\, dt \underset{\substack{at=\tau \\ t=\frac{1}{a}\tau}}{=} \int f(\tau)\, e^{-j\left(\frac{\omega}{a}\right)\tau} \left(\frac{1}{a}\right) d\tau$$

$$= \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

example

$$f(-t) \leftrightarrow \frac{1}{|-1|} F\left(\frac{\omega}{-1}\right) = F(-\omega)$$

$$f(at - t_0) = f\left(a\left(t - \frac{t_0}{a}\right)\right) \longleftrightarrow$$

$$\frac{1}{|a|} F\left(\frac{\omega}{a}\right) e^{-j t_0 \left(\frac{\omega}{a}\right)}$$