EE103 Lect #9 Oct 18, 2017

Quiz 1 Average = 7.15/10.0 $\sigma = 2.02$ (88 students)

chap 4.1 – 4.2

HW #3 (4 problems) posted on the EE103 web site.

Quiz #3 on Monday, Oct 23

General Form of n_{th}-order Linear (ordinary) differential equation

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$
$$= b_m \frac{d^m x(t)}{dt^m} + b_{m-1}\frac{d^{m-1}x(t)}{dt^{m-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

where a_k, $k=0,n$ and b_j, $j=0,m$ are constants

More compact form

$$\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum_{j=0}^{m} b_j \frac{d^j x(t)}{dt^j} \quad (\mathrm{I})$$

solution $\boxed{y(t) = y_c(t) + y_p(t)}$

$y_c(t)$ = complementary solution (natural response)

$y_p(t)$ = particular solution (forced response)

Natural response = $y(t)\big|_{\text{when } x(t)=0}$, i.e. zero input

From (A) $\displaystyle\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = 0 \quad (\mathrm{II})$

In general $y_c(t) = C e^{st} \quad (\mathrm{III})$

From (II) & (III), we obtain

$$\left(\sum_{k=0}^{n} a_k s^k\right) C e^{st} = 0 \quad \Rightarrow$$

$$\Rightarrow \quad a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$$

$$a_n (s-s_1)(s-s_2)\cdots(s-s_n) = 0 \quad (\mathrm{IV})$$

then $y_c(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + \cdots + C_n e^{s_n t}$

(Example)

$$C\frac{dy(t)}{dt} = \frac{x(t) - y(t)}{R}$$
$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$
$$a_1 \frac{d^1 y(t)}{dt^1} + a_0 y(t) = x(t)$$
$$a_1 = RC,\ a_0 = 1,\ b_0 = 1$$

$y_c(t) = ?$ set $x(t) = 0$

$$RC\frac{dy}{dt} + y = 0$$

Let $y_c(t) = c_1 e^{s_1 t}$

$$RC\frac{d}{dt}\left(c_1 e^{s_1 t}\right) + c_1 e^{s_1 t} = 0$$
$$RC\, c_1 s_1 e^{s_1 t} + c_1 e^{s_1 t}$$
$$= (RC s_1 + 1) c_1 e^{s_1 t} = 0$$
$$\Rightarrow \boxed{RC s_1 + 1 = 0} \Rightarrow \boxed{s_1 = -\frac{1}{RC}}$$

$y_c(t) = c_1 e^{-\frac{1}{RC}t}$

$y(0) = 0$ $\quad \boxed{RC\frac{dy(t)}{dt} + y(t) = x(t)}$

For $x(t) = u(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$

$$RC\frac{dy_p(t)}{dt} + y_p(t) = 1,\ t>0$$

$y_p(t) = P$ (constant), $\frac{dy_p}{dt} = 0$

$y_p(t) = P = 1$

$$y(t) = y_c(t) + y_p(t) = c_1 e^{-\frac{1}{RC}t} + 1$$
$$y(0) = 0 = c_1 e^{-\frac{1}{RC}(0)} + 1 \Rightarrow c_1 = -1$$
$$\Rightarrow \boxed{y(t) = 1 - e^{-\frac{1}{RC}t}}$$

Revisit of BIBO stability

$$y(t) = \mathcal{H}\, x(t)$$
$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Taking absolute value of both sides

$$|y(t)| = \left|\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau\right|$$
$$\le \int_{-\infty}^{\infty} |x(\tau)|\, |h(t-\tau)|\, d\tau$$

$$\boxed{\begin{array}{l} |y(t)| \le N \text{ for all } |x(t)| \le M \quad (\text{B.I.B.O}) \\ \text{if and only if } \int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty \end{array}}$$

- $h(t) = u(t)\, A\, e^{-kt},\ k>0$ BIBO
- $h(t) = u(t)$ Integrator, not BIBO
- $h(t) = R(t)$ not BIBO
- $h(t) = rect(t - \frac{T}{2}|T)$ BIBO
- $h(t) = \sum_{k=0}^{\infty} \delta(t-kT)$ not BIBO

Joseph Fourier

Jean-Baptiste Joseph Fourier

Jean Baptiste Joseph Fourier (born March 21, 1768, in Auxerre, Bourgogne, France; died May 16, 1830.) A mathematician known also as an Egyptologist and administrator, he exerted strong influence on mathematical physics through his *Théorie analytique de la chaleur* (1822; *The Analytical Theory of Heat*). He showed how the conduction of heat in solid bodies may be analyzed in terms of infinite mathematical series now called by his name, the Fourier series. Far transcending the particular subject of heat conduction, his work stimulated research in mathematical physics.

Reference: "Fourier, Joseph, Baron." Encyclopedia Britannica, 2007. Encyclopedia Britannica Online, January 6, 2007: http://www.britannica.com/eb/article-9035044

Chapter 4 Fourier Series

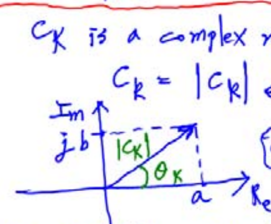
TABLE 4.2 Forms of the Fourier Series

Name	Equation
Exponential	$\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$; $C_k = \lvert C_k\rvert e^{j\theta_k}, C_{-k} = C_k^*$
Combined trigonometric	$C_0 + \sum_{k=1}^{\infty} 2\lvert C_k\rvert \cos(k\omega_0 t + \theta_k)$
Trigonometric	$A_0 + \sum_{k=1}^{\infty}(A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$ $2C_k = A_k - jB_k, C_0 = A_0$
Coefficients	$C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k\, e^{jk\omega_0 t} \quad (1)$$

C_k is a complex number

$$C_k = |C_k|\, e^{j\theta_k} \quad (2)$$



$$\boxed{C_k = a + jb}$$

$$\theta_k = \tan^{-1}\frac{b}{a}$$

From (1) & (2), $x(t) = \sum_{k=-\infty}^{\infty} |C_k|\, e^{j(k\omega_0 t + \theta_k)}$

(3)

$$= \underbrace{|C_0|\, e^{j(\theta_0)}}_{=C_0} + \sum_{k=1}^{\infty}\left(C_{-k}e^{-jk\omega_0 t} + C_{+k}e^{+jk\omega_0 t}\right) \quad (4)$$

$$\text{where } C_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt \quad (5)$$

$$C_{-k} = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-j(-k)\omega_0 t}\, dt \quad (6)$$

$$C_k^* = \frac{1}{T_0}\int_{T_0} x(t)\, e^{+jk\omega_0 t} = C_{-k} \quad (7)$$

$$|C_k| = |C_{-k}| = \frac{1}{T_0}\int_{T_0} |x(t)|\, dt \quad (8)$$

$$\theta_{-k} = -\theta_k \quad (9)$$

$$\Rightarrow C_{-k} = |C_{-k}|\, e^{+j\theta_{-k}} \underset{(8),(9)}{=} |C_k|\, e^{-j\theta_k} = (C_k)^* \quad (10)$$

thus from (4) & (10

$$x(t) = C_0 + \sum_{k=1}^{\infty} |C_k|\left[e^{-j(k\omega_0 t+\theta_k)} + e^{+j(k\omega_0 t+\theta_k)}\right]$$

$$\left(e^{j\phi}+e^{-j\phi} = 2\cos\phi\right) \quad = C_0 + \sum_{k=1}^{\infty} 2|C_k|\cos(k\omega_0 t+\theta_k) \quad (11)$$

$$\text{From (11)}\quad x(t) = C_0 + \sum_{k=1}^{\infty} 2|C_k|\cos(k\omega_0 t+\theta_k)$$

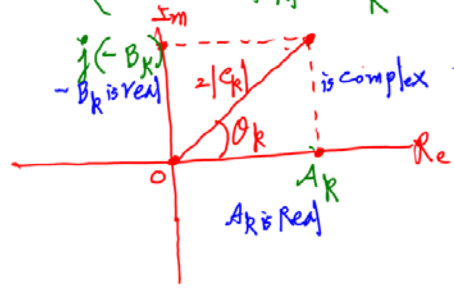
$$= C_0 + \sum_{k=1}^{\infty}\left[2|C_k|(\cos k\omega_0 t \cos\theta_k) - 2|C_k|\sin k\omega_0 t \sin\theta_k\right]$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= C_0 + \sum_{k=1}^{\infty}\left[2|C_k|\cos\theta_k \cos k\omega_0 t + (-2|C_k|\sin\theta_k \sin k\omega_0 t)\right]$$

$$= C_0 + \sum_{k=1}^{\infty}\left(A_k\cos k\omega_0 t + B_k \sin k\omega_0 t\right) \quad (12)$$

$$\text{where}\begin{cases} A_k = 2|C_k|\cos\theta_k \\ B_k = -2|C_k|\sin\theta_k \end{cases}$$



Calculation of C_n

$$C_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{T_0}\left(\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}\right)e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{T_0}\sum_{k=-\infty}^{\infty} C_k e^{j(k-n)\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{T_0} C_n e^{j(n-n)\omega_0 t}\, dt \quad (k=n)$$

$$+ \frac{1}{T_0}\int_{T_0}\sum_{k=-\infty}^{\infty} C_k e^{j(k-n)\omega_0 t}\, dt \quad (k \neq n)$$

$$= \frac{1}{T_0}\int_{T_0} C_n e^{0}\, dt + \sum_{\substack{n=-\infty \\ n\neq k}}^{\infty} \underbrace{\int_{T_0} C_k \overbrace{e^{+j(k-n)\omega_0 t}}^{\text{periodic}}\, dt}_{=0}$$

$$= \frac{1}{\cancel{T_0}} C_n \cancel{T_0} + 0 = C_n$$

$$\boxed{C_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt}$$

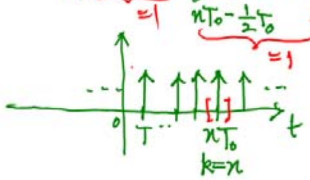
Example 1 $\quad x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT_0)$

$$C_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{T_0} \delta(t - nT_0)\, e^{-jn\omega_0 t}\, dt$$

$$+ \frac{1}{T_0}\int_{T_0} \sum_{\substack{k=-\infty \\ k\neq n}}^{\infty} \delta(t-kT_0)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T_0} \underbrace{e^{-jn\omega_0 (nT_0)}}_{=1} \underbrace{\int_{nT_0-\frac{1}{2}T_0}^{nT_0+\frac{1}{2}T_0} \delta(t-nT_0)\, dt}_{=1} + \underbrace{\frac{1}{T_0}\int_{nT_0-\frac{1}{2}T_0}^{nT_0+\frac{1}{2}T_0} \sum \cdots \, dt}_{=0}$$



$$\underline{= \frac{1}{T_0} + 0}$$