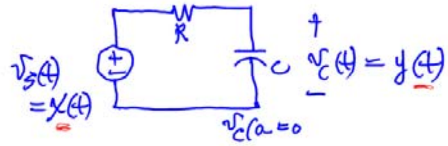
EE103 Lect #7 Oct 13, 2017

Note: Course materials (lecture slides, HWs & solutions) are posted on EE103 website:
https://ee103-fall2017-01.courses.soe.ucsc.edu/

Also webcast can be watched at https://webcast.ucsc.edu & EE103

Let us consider a simple RC circuit

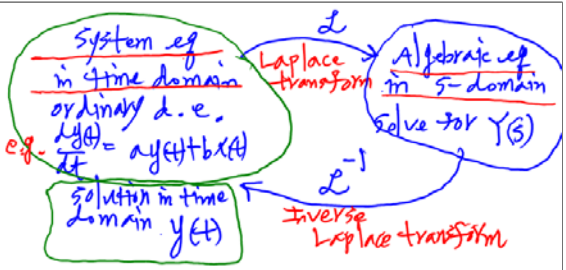


$$i_C(t) = C\frac{dv_C(t)}{dt} = C\frac{dy(t)}{dt} \quad (1)$$

$$i_C(t) = i_R(t) = \frac{v_S(t)-v_C(t)}{R} = \frac{x(t)-y(t)}{R} \quad (2)$$

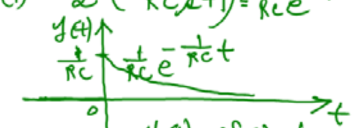
From (1) & (2)

$$C\frac{dy}{dt} = \frac{1}{R}(x-y) \quad \text{or} \quad \boxed{RC\frac{dy}{dt} = x-y} \quad (3)$$

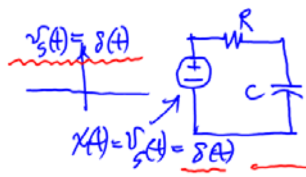


Taking Inverse Laplace Transform of (6) →

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{RCs+1}\right) = \frac{1}{RC}e^{-\frac{1}{RC}t}u(t)$$

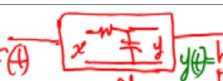


$$y(t) = v_C(t) = \begin{cases} 0 & t<0 \\ \frac{e^{-\frac{1}{RC}t}}{RC} & t \geq 0 \end{cases}$$



$$y(t) = \frac{1}{RC}u(t)e^{-\frac{1}{RC}t}$$

Impulse Response $h(t)$

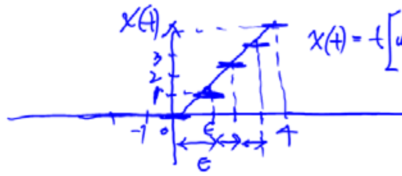


$x(t) = \delta(t)$ → $y(t) = h(t) = \frac{1}{RC}u(t)e^{-\frac{1}{RC}t}$ Impulse Response

For arbitrary $x(t)$ what would be $y(t) = ?$

(Ans) $y(t) = x(t) * h(t)$

WHY?



$x(t) = t[u(t) - u(t-4)]$

$$\tilde{x}(t) = \underset{0}{x(0)}\,\text{rect}(t|\epsilon) + x(\epsilon)\,\text{rect}(t-\epsilon|\epsilon)$$
$$+\, x(2\epsilon)\,\text{rect}(t-2\epsilon|\epsilon)$$
$$+\, x(3\epsilon)\,\text{rect}(t-3\epsilon|\epsilon)$$
$$+\, x(4\epsilon)\,\text{rect}(t-4\epsilon|\epsilon)$$

$$= \sum_{k=0}^{N} x(k\epsilon) \underbrace{\frac{1}{\epsilon} \mathrm{rect}(t - k\epsilon \,|\, \epsilon)}_{\delta(t-k\epsilon)} \epsilon$$

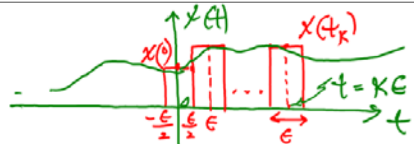
$$\hat{x}(t) = \lim_{\epsilon \to 0} \sum_{k=0}^{N\uparrow} x(k\epsilon) \frac{1}{\epsilon} \mathrm{rect}(t - k\epsilon \,|\, \epsilon) \epsilon$$

$k\epsilon = \tau$, $\epsilon = d\tau$, $N \to \infty$

$$\overset{\uparrow}{=} \int_{0^-}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t)\, \delta(t-\tau)\, d\tau$$

$$= x(t) \cdot 1 = x(t)$$



We note that $x(t)$ can be expressed as

$$x(t) = \lim_{\epsilon \to 0} \sum_{k=-\infty}^{\infty} x(k\epsilon) \frac{1}{\epsilon} \mathrm{rect}(t - k\epsilon \,|\, \epsilon)\, \epsilon$$

$$= \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

Also $y(t) = \mathbf{H}\, x(t)$ $\quad x \to \boxed{H} \to y$

$$= \mathcal{H}\left[\lim_{\epsilon \to 0} \sum_{k=-\infty}^{\infty} x(k\epsilon)\, \mathrm{rect}(t - k\epsilon \,|\, \epsilon) \right]$$

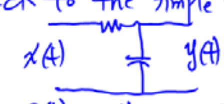
$$= \lim_{\epsilon \to 0} \sum_{k=-\infty}^{\infty} x(k\epsilon)\, \mathbf{H} \frac{1}{\epsilon} \mathrm{rect}(t - k\epsilon \,|\, \epsilon)\, \epsilon$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= x(t) * h(t) = h(t) * x(t)$$

convolution

Back to the simple RC circuit



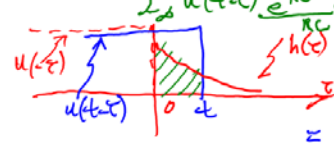
$\delta(t) \xrightarrow{\mathcal{H}} h(t)$

$h(t) = \frac{1}{RC} u(t) e^{-\frac{1}{RC} t}$

what is $y(t)$ for $x(t) = u(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} \frac{u(\tau)}{RC} e^{-\frac{1}{RC}(t-\tau)} \cdot u(t-\tau)\, d\tau$$

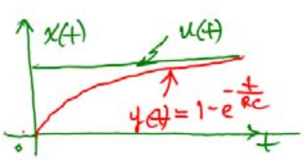
$$= \int_{-\infty}^{\infty} u(t-\tau) \frac{e^{-\frac{1}{RC}\tau}}{RC} u(\tau)\, d\tau$$

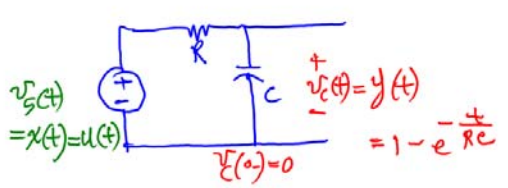


$$= \int_0^t \frac{1}{RC} e^{-\frac{1}{RC}\tau}\, d\tau$$

$$= \int_0^t d\left[-e^{-\frac{1}{RC}\tau}\right]$$

$$= \left(1 - e^{-\frac{1}{RC} t}\right) = y(t)$$





Convolution of periodic signal

2.7. Let $h(t)$ be the triangular pulse shown in Fig. 2-10(a) and let $x(t)$ be the unit impulse train [Fig. 2-10(b)] expressed as

$$x(t) = \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \qquad (2.68)$$

Determine and sketch $y(t) = h(t) * x(t)$ for the following values of T: (a) $T = 3$, (b) $T = 2$, (c) $T = 1.5$.

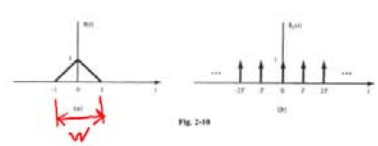


Fig. 2-10

$$y(t) = h(t) * \delta_T(t) = h(t) * \left[\sum_{n=-\infty}^{\infty} \delta(t - nT) \right]$$

$$= \sum_{n=-\infty}^{\infty} h(t) * \delta(t - nT) = \sum_{n=-\infty}^{\infty} h(t - nT) \qquad (2.69)$$

(a) For $T = 3$, Eq. (2.69) becomes

$$y(t) = \sum_{n=-\infty}^{\infty} h(t - 3n) \qquad T > W$$

which is sketched in Fig. 2-11(a).

(b) For $T = 2$, Eq. (2.69) becomes

$$y(t) = \sum_{n=-\infty}^{\infty} h(t - 2n) \qquad T = W$$

which is sketched in Fig. 2-11(b).

(c) For $T = 1.5$, Eq. (2.69) becomes

$$y(t) = \sum_{n=-\infty}^{\infty} h(t - 1.5n) \qquad T < W$$

which is sketched in Fig. 2-11(c). Note that when $T < 2$, the triangular pulses are no longer separated and they overlap.

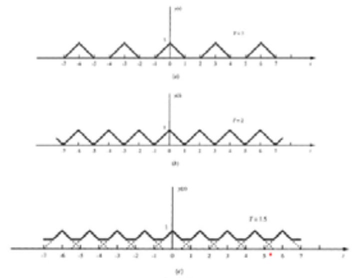


Fig. 2-11

Properties of Convolution

$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$

$\delta(t) \xrightarrow{H} h(t)$ impulse response

$x(t) \xrightarrow{H} y(t) = x(t) * h(t)$

Property 1 (commutative)

$$x(t) * h(t) = h(t) * x(t)$$

$$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Property 2 (Associative)

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

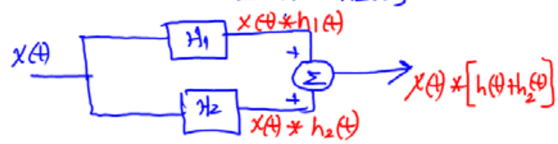
$x(t) \longrightarrow \boxed{H_1} \xrightarrow{x * h_1} \boxed{H_2} \longrightarrow (x * h_1) * h_2$

$\parallel$

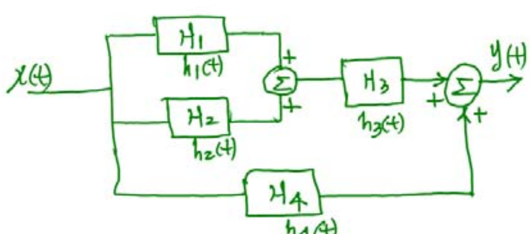
$x(t) \longrightarrow \boxed{H_S} \longrightarrow x(t) * (h_1(t) * h_2(t))$

Property 3 (Distributive)

$$x(t) * h_1(t) + x(t) * h_2(t)$$
$$= x(t) * [h_1(t) + h_2(t)]$$



$x(t) \longrightarrow \boxed{H_p} \longrightarrow x(t) * [h_1(t) + h_2(t)]$



$$y(t) = x(t) * h_4(t) + [x(t) * (h_1(t) + h_2(t))] * h_3(t)$$

$$= x(t) * [(h_1(t) + h_2(t)) * h_3(t) + h_4(t)]$$

General Form of nth-order Linear (ordinary) differential equation

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_m \frac{d^m x(t)}{dt^m} + b_{m-1} \frac{d^{m-1} x(t)}{dt^{m-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

where a_k, $k = 0, n$ and b_j, $j = 0, m$ are constants

More compact form

$$\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum_{j=0}^{m} b_j \frac{d^j x(t)}{dt^j} \qquad (I)$$

solution $y(t) = y_c(t) + y_p(t)$

$y_c(t)$ = complementary solution (natural response)

$y_p(t)$ = particular solution (forced response)

Natural response $= y(t)\big|_{\text{when } x(t)=0}$, i.e. zero input

From (A) $\displaystyle\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = 0$ (II)

In general $y_c(t) = C e^{st}$ (III)

From (II) & (III), we obtain

$\left(\displaystyle\sum_{k=0}^{n} a_k s^k\right) C e^{st} = 0 \Rightarrow$

$\Rightarrow a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$

$a_n (s-s_1)(s-s_2)\cdots(s-s_n) = 0$ (IV)

then $y_c(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + \cdots + c_n e^{s_n t}$

(Example)

$C\dfrac{dy(t)}{dt} = \dfrac{x(t) - y(t)}{R}$

$RC\dfrac{dy(t)}{dt} + y(t) = x(t)$

$a_1 \dfrac{d^1 y(t)}{dt^1} + a_0 y(t) = x(t)$

$a_1 = RC,\ a_0 = 1,\ b_0 = 1$

$y_c(t) = ?$ set $x(t) = 0$

$RC\dfrac{dy}{dt} + y = 0$

Let $y_c(t) = c_1 e^{s_1 t}$

$RC\dfrac{d}{dt}\left(c_1 e^{s_1 t}\right) + c_1 e^{s_1 t} = 0$

$RC\, c_1 s_1 e^{s_1 t} + c_1 e^{s_1 t}$

$= (RC\, s_1 + 1)\, c_1 e^{s_1 t} = 0$

$\Rightarrow RC\, s_1 + 1 = 0 \Rightarrow s_1 = -\dfrac{1}{RC}$

$y_c(t) = c_1 e^{-\frac{1}{RC}t}$

$y(0) = 0$ $\quad RC\dfrac{dy(t)}{dt} + y(t) = x(t)$

For $x(t) = u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

$RC\dfrac{dy_p(t)}{dt} + y_p(t) = 1,\ t > 0$

$y_p(t) = P$ (constant), $\dfrac{dy_p}{dt} = 0$

$y_p(t) = P = 1$

$y(t) = y_c(t) + y_p(t) = c_1 e^{-\frac{1}{RC}t} + 1$

$y(0) = 0 = c_1 e^{-\frac{1}{RC}(0)} + 1 \Rightarrow c_1 = -1$

$\Rightarrow \boxed{y(t) = 1 - e^{-\frac{1}{RC}t}}$

Revisit of BIBO stability

$y(t) = \mathcal{H}\, x(t)$

$= \displaystyle\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

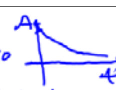
Taking absolute value of both sides

$|y(t)| = \left|\displaystyle\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau\right|$

$\le \displaystyle\int_{-\infty}^{\infty} |x(\tau)|\, |h(t-\tau)|\, d\tau$

$\boxed{\begin{array}{l} |y(t)| \le N \text{ for all } |x(t)| \le M \quad \text{(B.I.B.O)} \\ \text{if and only if } \int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty \end{array}}$

- $h(t) = A e^{-kt},\ k > 0$ BIBO



- $h(t) = u(t)$ Integrator, not BIBO
- $h(t) = R(t)$ not BIBO

- $h(t) = \text{rect}\left(t - \frac{T}{2} \middle| T\right)$ BIBO

- $h(t) = \displaystyle\sum_{k=0}^{\infty} \delta(t - kT)$ not BIBO