EE 103 Lect #11 Oct. 23, 2017

HW #4 posted

Quiz #3 Today; Midterm on Nov. 1

Quiz #2 Avg = 6.67 σ = 1.21

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$$

$$\delta(t) \xrightarrow{H} h(t)$$

$$\text{Periodic } x(t) \xrightarrow{H} \text{periodic } y(t)$$

$$(FS)\quad x(t) = \sum_{k=-\infty}^{\infty} C_{kx} e^{+jk\omega_0 t} \xrightarrow{H} y(t) = \sum_{k=-\infty}^{\infty} C_{ky} e^{+jk\omega_0 t}$$

For a particular k (kth term)

$$C_{kx} e^{+jk\omega_0 t} \xrightarrow{H} C_{ky} e^{+jk\omega_0 t}$$

LTI does not alter frequency

Fourier Series

TABLE 4.2 Forms of the Fourier Series

Name	Equation
Exponential	$\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$; $C_k = \lvert C_k\rvert e^{j\theta_k}, C_{-k} = C_k^*$
Combined trigonometric	$C_0 + \sum_{k=1}^{\infty} 2\lvert C_k\rvert \cos(k\omega_0 t + \theta_k)$
Trigonometric	$A_0 + \sum_{k=1}^{\infty}(A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$ $2C_k = A_k - jB_k, C_0 = A_0$
Coefficients	$C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$

For a periodic signal $x(t)$ with its minimum period T_0, $x(t+T_0) = x(t)$ for all t.

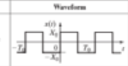
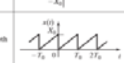
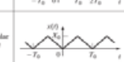
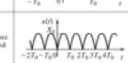
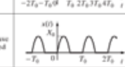
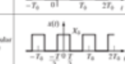
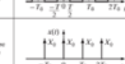
$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \qquad \left(\omega_0 = \frac{2\pi}{T_0}\right)$$

$$\boxed{C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt}$$

Alternatively

$$C_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

TABLE 4.3 Fourier Series for Common Signals

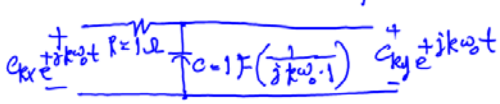
Name	Waveform	C_0	$C_k, k \neq 0$	Comments
1. Square wave		0	$-j\frac{2X_0}{\pi k}$	$C_k = 0$, k even
2. Sawtooth		$\frac{X_0}{2}$	$j\frac{X_0}{2\pi k}$	
3. Triangular wave		$\frac{X_0}{2}$	$\frac{-2X_0}{(\pi k)^2}$	$C_k = 0$, k even
4. Full-wave rectified		$\frac{2X_0}{\pi}$	$\frac{-2X_0}{\pi(4k^2-1)}$	
5. Half-wave rectified		$\frac{X_0}{\pi}$	$\frac{-X_0}{\pi(k^2-1)}$	$C_k = 0$, k odd, except $C_1 = -j\frac{X_0}{4}$ and $C_{-1} = j\frac{X_0}{4}$
6. Rectangular wave		$\frac{TX_0}{T_0}$	$\frac{TX_0}{T_0}\operatorname{sinc}\frac{Tk\omega_0}{2}$	$\frac{Tk\omega_0}{2} = \frac{\pi Tk}{T_0}$
7. Impulse train		$\frac{X_0}{T_0}$	$\frac{X_0}{T_0}$	

see EX 4.2 on page 165

See EX 4.6 on page 171

covered in Lect #9

Back to the RC circuit example



$$C_{ky} = \frac{\frac{1}{jk\omega_0}}{1 + \frac{1}{jk\omega_0}} C_{kx}$$

$$= \frac{1}{1 + j(k\omega_0)} C_{kx}$$

$$\Rightarrow C_{ky} = \frac{1}{\sqrt{1 + (k\omega_0)^2}} e^{-j\tan^{-1}(k\omega_0)} C_{kx}$$

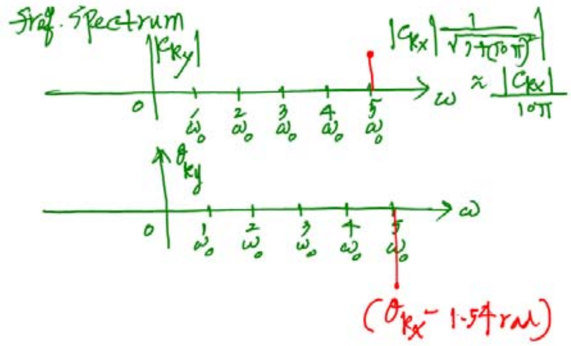
$$\lvert C_{ky}\rvert = \frac{1}{\sqrt{1 + (k\omega_0)^2}} \lvert C_{kx}\rvert$$

$$\theta_{ky} = \theta_{kx} - \tan^{-1} k\omega_0$$

If $T_0 = 1$ (and $\omega_0 = \frac{2\pi}{T_0} = 2\pi$); $k = 5$

$$\lvert C_{5y}\rvert = \frac{1}{\sqrt{1 + (5(2\pi))^2}} \lvert C_{5x}\rvert$$

$$\theta_{ky} = \theta_{kx} - \tan^{-1}(5(2\pi)) = -1.54 \text{ rad}$$
$$= -88.24^\circ$$

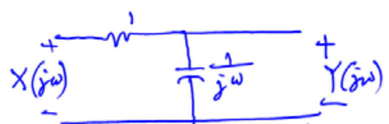


In general, for $x(t)$ periodic with period T_0 (fundamental freq ω_0) $y(t) = H\,x(t)$ can be represented in frequency domain as

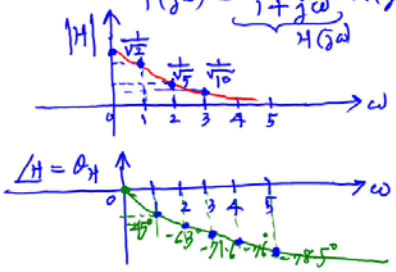
$$\sum_{k=-\infty}^{\infty} C_{k_x} e^{+jk\omega_0 t} \longrightarrow \boxed{H(jk\omega_0)} \longrightarrow \sum_{k=-\infty}^{\infty} C_{k_y} e^{+jk\omega_0 t}$$

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

$$Y(j\omega) = H(j\omega)\,X(j\omega)$$



$$Y(j\omega) = \underbrace{\frac{1}{1+j\omega}}_{H(j\omega)} X(j\omega)$$



when $y(t) = x(-t)$ with period T_0 (& freq ω_0)

FS $\quad y(t) = \underline{x(-t)} = \sum_{k=-\infty}^{\infty} C_{k_x} e^{+jk\omega_0(-t)}$

$$= \sum_{k=-\infty}^{\infty} C_{k_x} e^{+j(-k)\omega_0 t}$$

$$\overset{\hat{k} \leftarrow -k}{=} \sum_{\hat{k}=\infty}^{-\infty} C_{-\hat{k}_x} e^{+j\hat{k}\omega_0 t}$$

$$= \sum_{\hat{k}=-\infty}^{\infty} C_{-\hat{k}_x} e^{+j\hat{k}\omega_0 t} = \sum_{n=-\infty}^{\infty} C_{-n_x} e^{+jn\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} C_{k_y} e^{+jk\omega_0 t} = \sum_{k=-\infty}^{\infty} C_{-k_x} e^{+jk\omega_0 t} = \sum_{k=-\infty}^{\infty} C^*_{k_x} e^{+jk\omega_0 t}$$

Note = when time is reversed $(t \to -t)$ k-th coefficient becomes complex conjugate

$$C_{k_y} = C^*_{k_x} \;(= C_{-k_x})$$

with same magnitude but k-th phase angle sign is reversed.

Summary

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t) = x(t) * h(t)$$

$$X(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s) = H(s)X(s)$$

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

For $x(t) = x(t + T_0)$ for all t; period T_0 and freq. $\omega_0 = \frac{2\pi}{T_0}$.

$$c_{kx} = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{2\pi}\,\frac{2\pi}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$\lim_{T_0\to\infty} c_{kx} = \frac{1}{2\pi}\lim_{T_0\to\infty}\frac{2\pi}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{2\pi}\, d\omega \underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}_{\text{Fourier Transform } \mathcal{F}}$$

$$= \frac{1}{2\pi}\left[X(\omega)\right] d\omega$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_{kx}\, e^{+jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty}\left[\frac{1}{2\pi}X(\omega)\, d\omega\right] e^{jk\omega_0 t}$$

$\omega_0 = \frac{2\pi}{T_0} = d\omega$

$k\omega_0 = \omega$

$\Sigma \to \int$

$$= \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega}_{\text{Inverse Fourier Transform}}$$

$$\boxed{\begin{aligned}\mathcal{F}[x(t)] &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = X(\omega)\\ \mathcal{F}^{-1}[X(\omega)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega\end{aligned}}$$

Consider $x(t) = X_0\, \text{rect}(t/T)$

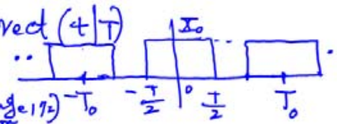


Table 4.3 case 6 (Page 172)

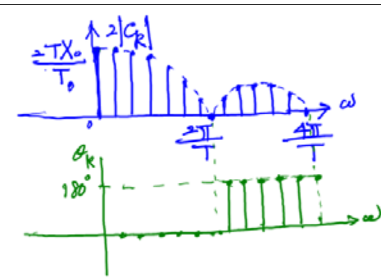
$c_0 = \frac{TX_0}{T_0}$

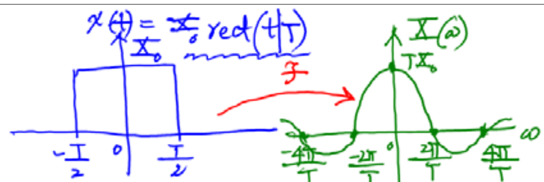
$k \neq 0 \quad c_k = \frac{TX_0}{T_0}\,\text{sinc}\left(\frac{Tk\omega_0}{2}\right)$

$x(t) = c_0 + \sum_{k=1}^{\infty} 2|c_k| \cos(k\omega_0 t + \theta_k)$ (Ref. Table 4.2)

$2|c_k| = \frac{2TX_0}{T_0}\,\text{sinc}\,\frac{Tk\omega_0}{2}$ $\quad \text{sinc}\, x = \frac{\sin x}{x}$

$$\theta_k = \begin{cases} 0^\circ, & \text{sinc}(Tk\omega_0/2) > 0 \\ 180^\circ, & \text{sinc}(Tk\omega_0/2) < 0 \end{cases}$$





$X(\omega) = X_0 T\, \text{sinc}\,\frac{\omega T}{2}$

$$X(\omega) = \int x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T/2}^{T/2} X_0\, e^{-j\omega t}\, dt$$

$$= X_0\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\frac{T}{2}}^{\frac{T}{2}} = 2X_0\,\frac{e^{j\frac{\omega T}{2}} - e^{-j\frac{\omega T}{2}}}{2j\omega} = 2X_0\,\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\,\frac{T}{2}$$

$$= \boxed{X_0 T\, \text{sinc}(\omega T/2)}$$