EE103 Lect #10 Oct 20, 201

HW #3 (4 problems) posted on the EE103 website.

Quiz #3 on Monday, Oct 23

Midterm on Wednesday, Nov. 1.

Fourier Series

TABLE 4.2 Forms of the Fourier Series

Name	Equation
Exponential	$\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$; $C_k = \lvert C_k\rvert e^{j\theta_k}, C_{-k} = C_k^*$
Combined trigonometric	$C_0 + \sum_{k=1}^{\infty} 2\lvert C_k\rvert \cos(k\omega_0 t + \theta_k)$
Trigonometric	$A_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$ $2C_k = A_k - jB_k, C_0 = A_0$
Coefficients	$C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$

For a periodic signal $x(t)$ with its minimum period T_0, $x(t+T_0) = x(t)$ for all t.

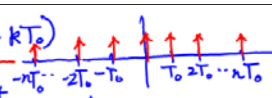
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad \left(\omega_0 = \frac{2\pi}{T_0}\right)$$

$$\boxed{c_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt}$$

Alternatively

$$c_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

For $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$



$$c_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0} \text{ for any } n.$$

$$\Rightarrow x(t) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

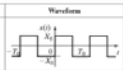
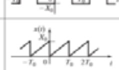
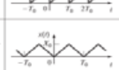
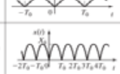
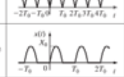
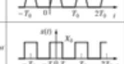
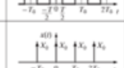
$$= \frac{1}{T_0}\left(1 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} e^{jk\omega_0 t}\right)$$

$$= \frac{1}{T_0}\left[1 + \sum_{k=1}^{\infty}\left(e^{-jk\omega_0 t} + e^{+jk\omega_0 t}\right)\right]$$

$$= \frac{1}{T_0}\left[1 + \sum_{k=1}^{\infty} 2\cos k\omega_0 t\right]$$

$e^{j\theta} + e^{-j\theta} = 2\cos\theta$

TABLE 4.3 Fourier Series for Common Signals

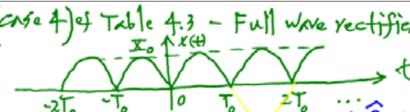
Name	Waveform	C_0	$C_k, k \neq 0$	Comments
1. Square wave		0	$-j\frac{2X_0}{\pi k}$	$C_k = 0$, k even
2. Sawtooth		$\frac{X_0}{2}$	$j\frac{X_0}{2\pi k}$	
3. Triangular wave		$\frac{X_0}{2}$	$\frac{-2X_0}{(\pi k)^2}$	$C_k = 0$, k even
4. Full-wave rectified		$\frac{2X_0}{\pi}$	$\frac{-2X_0}{\pi(4k^2-1)}$	
5. Half-wave rectified		$\frac{X_0}{\pi}$	$\frac{-X_0}{\pi(k^2-1)}$	$C_k = 0$, k odd, except $C_1 = -j\frac{X_0}{4}$ and $C_{-1} = j\frac{X_0}{4}$
6. Rectangular wave		$\frac{TX_0}{T_0}$	$\frac{TX_0}{T_0}\text{sinc}\frac{Tk\omega_0}{2}$	$\frac{Tk\omega_0}{2} = \frac{\pi Tk}{T_0}$
7. Impulse train		$\frac{X_0}{T_0}$	$\frac{X_0}{T_0}$	

see EX 4.2 on page 165

See EX 4.6 on page 171

covered in Lect #9

(Case 4) of Table 4.3 – Full wave rectified



$x(t) = \lvert X_0 \sin \hat{\omega}_0 t \rvert$ $\quad$ ($\hat{\omega}_0 2T_0 = 2\pi$, $\omega_0 T_0 = 2\pi$, $\omega_0 = 2\hat{\omega}_0$

Fourier Series (FS) $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t}$

where $c_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$

$$= \frac{1}{T_0}\int_0^{T_0} X_0 \sin \hat{\omega}_0 t \, e^{-jk\omega_0 t} dt$$

$$= \frac{X_0}{T_0}\int_0^{T_0} \frac{e^{+j\hat{\omega}_0 t} - e^{-j\hat{\omega}_0 t}}{2j} e^{-jk\omega_0 t} dt$$

$$= \frac{X_0}{T_0}\int_0^{T_0} \frac{e^{+j\frac{\omega_0}{2}t} - e^{-j\frac{\omega_0}{2}t}}{2j}\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{X_0}{2jT_0}\int_0^{T_0}\left[e^{-j\omega_0(k-\frac{1}{2})t} - e^{-j\omega_0(k+\frac{1}{2})t}\right]dt$$

$$= \frac{X_0}{2jT_0}\left[\frac{e^{-j\omega_0(k-\frac{1}{2})t}}{-j\omega_0(k-\frac{1}{2})} - \frac{e^{-j\omega_0(k+\frac{1}{2})t}}{-j\omega_0(k+\frac{1}{2})}\right]_0^{T_0}$$

$$= \frac{X_0}{2j^2\omega_0T_0}\left[\frac{e^{-j\omega_0T_0(k+\frac{1}{2})}-1}{k+\frac{1}{2}} - \frac{e^{-j\omega_0T_0(k-\frac{1}{2})}-1}{k-\frac{1}{2}}\right]$$

$$= \frac{X_0}{(-1)4\pi}\left[\frac{e^{-j2\pi(k+\frac{1}{2})}-1}{k+\frac{1}{2}} - \frac{e^{-j2\pi(k-\frac{1}{2})}-1}{k-\frac{1}{2}}\right]$$

For $k=0$

$$c_0 = \frac{X_0}{-4\pi}\left(\frac{e^{-j2\pi(0+\frac{1}{2})}-1}{0+\frac{1}{2}} - \frac{e^{-j2\pi(0-\frac{1}{2})}-1}{0-\frac{1}{2}}\right)$$

$$= \frac{X_0}{-4\pi}\left(\underbrace{\frac{-1-1}{\frac{1}{2}}}_{=-4} - \underbrace{\frac{-1-1}{-\frac{1}{2}}}_{=4}\right) = \boxed{\frac{2X_0}{\pi}}$$

$c_0 = \frac{2X_0}{\pi}$ (DC value of $x(t)$ Average)

For $k \neq 0$

$$c_k = \frac{X_0}{-4\pi}\left(\frac{e^{-j2\pi(k+\frac{1}{2})}-1}{k+1} - \frac{e^{-j2\pi(k-\frac{1}{2})}-1}{k-1}\right)$$

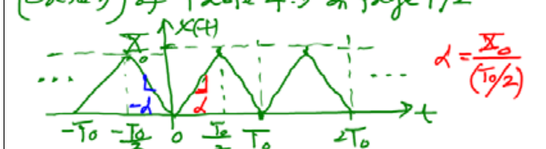
$$= \frac{X_0}{-4\pi}\left(\frac{-e^{-j2\pi k}-1}{k+1} - \frac{-e^{-j2\pi k}-1}{k-1}\right)$$

$$= \frac{X_0}{-4\pi}\left(\frac{-2}{k+\frac{1}{2}} - \frac{-2}{k-\frac{1}{2}}\right)$$

$$= \frac{X_0}{2\pi}\left(\underbrace{\frac{1}{k+\frac{1}{2}} - \frac{1}{k-\frac{1}{2}}}_{=\frac{-1}{k^2-\frac{1}{4}} = \frac{-4}{4k^2-1}}\right)$$

$$= \boxed{\frac{-2X_0}{\pi(4k^2-1)} = c_k}, \quad k \neq 0 \qquad ⊠$$

(Case 3) of Table 4.3 on page 172



FS: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t}$, $\omega_0 = \frac{2\pi}{T_0}$

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\left[\underbrace{\int_{-\frac{T_0}{2}}^{0} (-\alpha t)\, e^{-jk\omega_0 t}\, dt}_{\text{for this part } t \to -\tau} + \int_0^{\frac{T_0}{2}} (\alpha t)\, e^{-jk\omega_0 t}\, dt\right]$$

$$= \frac{1}{T_0}\left[\int_{+\frac{T_0}{2}}^{0} \alpha\tau\, e^{+jk\omega_0\tau}(-d\tau) + \int_0^{\frac{T_0}{2}} \alpha t\, e^{-jk\omega_0 t}\, dt\right]$$

$$= \frac{1}{T_0}\left[\int_0^{\frac{T_0}{2}} \alpha\tau\, e^{+jk\omega_0\tau}\, d\tau + \int_0^{\frac{T_0}{2}} \alpha t\, e^{-jk\omega_0 t}\, dt\right]$$

$$= \frac{1}{T_0}\int_0^{\frac{T_0}{2}} (\alpha t)\left[e^{+jk\omega_0 t} + e^{-jk\omega_0 t}\right] dt$$

$$= \frac{1}{T_0}\int_0^{\frac{T_0}{2}} (\alpha t)\left[2\cos k\omega_0 t\right] dt$$

For $k=0$ (DC case)

$$\frac{1}{T_0}\int_0^{\frac{T_0}{2}} (\alpha t)\big(\underbrace{2\cos(0)\omega_0 t}_{=2}\big)\, dt$$

$$= \frac{2}{T_0}\alpha\frac{t^2}{2}\Big|_0^{\frac{T_0}{2}} = \frac{2}{T_0}\left(\frac{X_0}{T_0/2}\right)\frac{\left(\frac{T_0}{2}\right)^2}{2}$$

$$= \frac{2X_0}{T_0^2}\cdot\frac{T_0^2}{4} = \boxed{\frac{X_0}{2}}$$

For $k \neq 0$

$$\frac{1}{T_0}\int_0^{\frac{T_0}{2}} \alpha t\; 2\cos k\omega_0 t\, dt$$

$$= \frac{2\alpha}{T_0}\int_0^{T_0/2} t\, d\left(\frac{\sin k\omega_0 t}{k\omega_0}\right) = \frac{2\alpha}{T_0}\left[t\,\frac{\sin k\omega_0 t}{k\omega_0} - \int \frac{\sin k\omega_0 t}{k\omega_0}\, dt\right]\Bigg|_0^{T_0/2}$$

$$= \frac{2\alpha}{T_0}\left[\frac{T_0}{2}\frac{\sin k\omega_0\left(\frac{T_0}{2}\right)}{k\omega_0} - 0 - \int_0^{T_0/2}\frac{\sin k\omega_0 t}{k\omega_0}\,dt\right]$$

$$\sin k\omega_0\left(\tfrac{T_0}{2}\right) = 0$$

$$= \frac{2\alpha}{T_0}\cdot\left.\frac{\cos k\omega_0 t}{(k\omega_0)^2}\right|_0^{\frac{T_0}{2}}$$

$$= 2\left(\frac{X_0}{T_0/2}\right)\frac{1}{T_0}\,\frac{\cos k\omega_0\left(\frac{T_0}{2}\right) - 1}{(k\omega_0)^2}$$

$$= 4X_0\frac{1}{T_0^2}\,\frac{\cos k\pi - 1}{(k\omega_0)^2}$$

$$= \begin{cases} \boxed{0} & \text{for even } k \\ 4X_0\frac{1}{T_0^2}\frac{-2}{(k\omega_0)^2} = \boxed{\frac{-2X_0}{(k\pi)^2}}, & \text{for odd } k \end{cases}$$

Properties of FS

A periodic function $x(t)$ that satisfies the following Dirichlet conditions can be expressed by a FS:

(1) at most a finite number of discontinuities in one period

(2) At most a finite number of maxima and minima in one period

(3) $x(t)$ is bounded, or $\int_{T_0}|x(t)|\,dt < \infty$ (may contain singularities)

also,

(i) $x(t)\big|_{t=t_a} = \sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t_a} = \frac{x(t_{a-}) + x(t_{a+})}{2}$

(ii) continuous function $x(t)$ can be uniformly approximated by

$$x_N(t) = \sum_{k=-N}^{N} c_k e^{+jk\omega_0 t} = c_0 + \sum_{k=1}^{N} 2|c_k|\cos(k\omega_0 t + \theta_k)$$

(iii) Error $\varepsilon(t) = x(t) - x_N(t)$

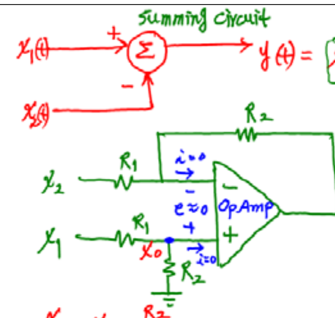
For $|\varepsilon(t)| < K$, N can be chosen

Mean-squared error $= \frac{1}{T_0}\int_{T_0}\varepsilon^2(t)\,dt$

(iv) For large k, $|c_k|$ decreases in proportion to $\frac{1}{k}$

(v) $FS(x_1(t) + x_2(t)) = FS\,x_1(t) + FS\,x_2(t)$

summing circuit



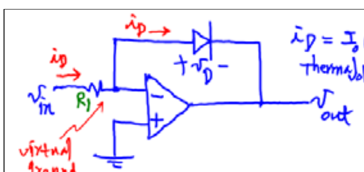
$y(t) = \boxed{x_1(t) - x_2(t)}$

$y(t) = \frac{R_2}{R_1}(x_1 - x_2) \underset{R_1 = R_2}{=} x_1 - x_2$

$x_0 = x_1\frac{R_2}{R_1 + R_2}$

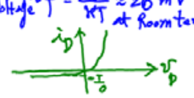
$y(t) = x_0 + R_2\frac{x_0 - x_2}{R_1} = x_0\left(1 + \frac{R_2}{R_1}\right) - \frac{R_2}{R_1}x_2$

$= x_1\frac{R_2}{R_1+R_2}\,\frac{R_1+R_2}{R_1} - \frac{R_2}{R_1}x_2 = \boxed{\frac{R_2}{R_1}(x_1 - x_2)}$



$i_D = I_0\left(e^{n v_D/V_T} - 1\right)$

thermal voltage $V_T = \frac{q}{KT} \approx 26\,mV$ at Room temp



$v_{in} = R_1 i_D = R_1 I_0\left(e^{n v_D/V_T} - 1\right), \quad -v_D = v_{out}$

$\approx R_1 I_0 e^{n v_D/V_T}$ for $v_D \gg V_T$

$= R_1 I_0 e^{-n v_{out}/V_T}$

$e^{-n v_{out}/V_T} = \frac{v_{in}}{R_1 I_0}$

$-n\frac{v_{out}}{V_T} = \ln\frac{v_{in}}{R_1 I_0} \Rightarrow \boxed{v_{out} = -\frac{1}{n}V_T \ln\frac{v_{in}}{R_1 I_0}}$