EE 103 Lect #13 Oct. 27, 2017

HW #4 posted

Quiz 4 on 10/30 ; Midterm on 11/1

$$x(t) \longrightarrow \boxed{H}_{\text{LTI}} \longrightarrow y(t)$$

$$\delta(t) \longrightarrow h(t)$$

$$\hat{x}(t) \longrightarrow h(t) * \hat{x}(t)$$

[Ref. chap. 5.3 ~ 5.4]

$$\text{(FS)}\; x(t) = \sum_{k=-\infty}^{\infty} C_{kx} e^{+jk\omega_0 t} \xrightarrow{\;H\;} y(t) = \sum_{k=-\infty}^{\infty} C_{ky} e^{+jk\omega_0 t}$$

T_0 period, ω_0

$T_0 \omega_0 = 2\pi$

$$\Rightarrow C_{ky} = H\, C_{kx} \quad \text{at } \omega = k\omega_0$$

$H = H(jk\omega_0)$ is a complex number dependent on $k\omega_0$ (input frequency)

Fourier Series

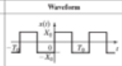
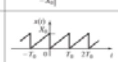
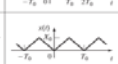
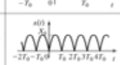
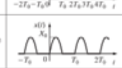
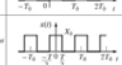
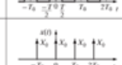
TABLE 4.2 Forms of the Fourier Series

Name	Equation
Exponential	$\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$; $C_k = \lvert C_k\rvert e^{j\theta_k}, C_{-k} = C_k^*$
Combined trigonometric	$C_0 + \sum_{k=1}^{\infty} 2\lvert C_k\rvert \cos(k\omega_0 t + \theta_k)$
Trigonometric	$A_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$ $2C_k = A_k - jB_k, C_0 = A_0$
Coefficients	$C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$

$$|C_{ky}| = |C_{kx}|\,|H|$$

$$\angle C_{ky} = \angle C_{kx} + \angle H \qquad \text{e.g. } \angle H = 40^\circ \;\; e^{j40^\circ}$$

TABLE 4.3 Fourier Series for Common Signals

Name	Waveform	C_0	$C_k, k \neq 0$	Comments
1. Square wave		0	$-j\frac{2X_0}{\pi k}$	$C_k = 0$, k even
2. Sawtooth		$\frac{X_0}{2}$	$j\frac{X_0}{2\pi k}$	
3. Triangular wave		$\frac{X_0}{2}$	$\frac{-2X_0}{(\pi k)^2}$	$C_k = 0$, k even
4. Full-wave rectified		$\frac{2X_0}{\pi}$	$\frac{-2X_0}{\pi(4k^2-1)}$	
5. Half-wave rectified		$\frac{X_0}{\pi}$	$\frac{-X_0}{\pi(k^2-1)}$	$C_k = 0$, k odd, except $C_1 = -j\frac{X_0}{4}$ and $C_{-1} = j\frac{X_0}{4}$
6. Rectangular wave		$\frac{TX_0}{T_0}$	$\frac{TX_0}{T_0}\text{sinc}\frac{Tk\omega_0}{2}$	$\frac{Tk\omega_0}{2} = \frac{\pi Tk}{T_0}$
7. Impulse train		$\frac{X_0}{T_0}$	$\frac{X_0}{T_0}$	

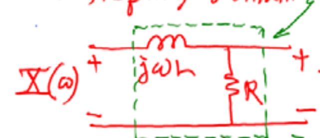
see EX 4.2 on page 165

See EX 4.6 on page 171

covered in Lect #9

A simple RL circuit

In frequency domain



$$Y(\omega) = H(\omega)X(\omega) = \underbrace{\left[\frac{R}{R + j\omega L}\right]}_{= H(\omega)} X(\omega)$$

$$X(\omega) = \mathcal{F}[x(t)]$$

From Table 4.2, a periodic signal $x(t)$ (T_0, ω_0) can be represented in the form of

1) exponential
$$\sum_{k=-\infty}^{\infty} C_k e^{+jk\omega_0 t}$$

2) combined trigonometric
$$C_0 + \sum_{k=1}^{\infty} |2C_k| \cos(k\omega_0 t + \theta_k)$$

3) Trigonometric
$$A_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$$

Consider the case of $x(t)$ expressed by (Trigonometric)

$$x(t) = A_0 + A_1 \cos \omega_0 t \qquad (1)$$

$$\mathcal{F}[x(t)] = \mathcal{F}[A_0 + A_1 \cos \omega_0 t]$$

$$= A_0 2\pi\delta(\omega) + A_1 \mathcal{F}[\cos \omega_0 t]$$

$$\mathcal{F}[\cos \omega_0 t] = \mathcal{F}\left[\frac{e^{+j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]$$

$$= \frac{1}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

Thus

$X(\omega) = A_0 2\pi\delta(\omega) + \frac{A_1}{2}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$

$H(\omega) = \frac{R}{R+j\omega L}$

$Y(\omega) = H(\omega)X(\omega) = \frac{R}{R+j\omega L}\left(A_0 2\pi\delta(\omega) + \frac{A_1}{2}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]\right)$

$= A_0 2\pi\delta(\omega) + \frac{A_1}{2}\left[\underbrace{\frac{R\,\delta(\omega-\omega_0)}{R+j\omega_0 L}}_{C_1} + \underbrace{\frac{R\,\delta(\omega+\omega_0)}{R-j\omega_0 L}}_{C_1^*}\right]$

$= A_0 2\pi\delta(\omega) + \frac{A_1}{2}\left(C_1\delta(\omega-\omega_0) + C_1^*\delta(\omega+\omega_0)\right)$

$\downarrow \mathcal{F}^{-1}$

$y(t) = A_0 + A_1\left(\frac{C_1 e^{j\omega_0 t} + C_1^* e^{-j\omega_0 t}}{2}\right)$

$= A_0 + A_1|C_1|\frac{e^{j(\omega_0 t+\theta_1)} + e^{-j(\omega_0 t+\theta_1)}}{2}$

$= A_0 + A_1|C_1|\cos(\omega_0 t + \theta_1)$

where $|C_1| = \left|\frac{R}{R+j\omega_0 L}\right| = \frac{R}{\sqrt{R^2+(\omega_0 L)^2}}$

$\theta_1 = \angle\frac{R}{R+j\omega_0 L} = 0 - \tan^{-1}\frac{\omega_0 L}{R}$

In Summary

For $x(t) = A_0 + A_1\cos\omega_0 t$ thru RL circuit

$y(t) = A_0 + A_1|H(\omega_0)|\cos(\omega_0 t + \angle H(\omega_0))$

$|H(\omega_0)| = \frac{R}{\sqrt{R^2+(\omega_0 L)^2}}$, $\angle H(\omega_0) = -\tan^{-1}\frac{\omega_0 L}{R}$

In general

$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$

$x(t)$ periodic T_0, ω_0; $h(t)$, $H(\omega)$, $H(s)$; $y(t)$ periodic T_0, ω_0

$x(t) = A_0 + \sum_{k=1}^{\infty} A_k\cos(k\omega_0 t + \phi_k)$

θ_k

$H(\omega)\big|_{\omega=k\omega_0} = |H(k\omega_0)|\angle H(k\omega_0)$

$y(t) = A_0 + \sum_{k=1}^{\infty} A_k|H(k\omega_0)|\cos(k\omega_0 t + \phi_k + \theta_k)$

Back to the RL circuit example

$H(\omega) = \frac{40}{40+j3\omega}$

For $x(t) = 10 + 5\cos \overset{\omega}{10} t$

$|H(\omega=10)| = \left|\frac{40}{40+j3(10)}\right| = \frac{40}{50} = \boxed{0.8}$

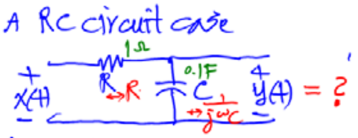
$\angle H(\omega=10) = 0 - \tan^{-1}\frac{30}{40} = \boxed{-36.9^\circ}$

$y(t) = 10 + 5(0.8)\cos(10t - 36.9^\circ)$

no change in DC

ans

A RC circuit case



$y(t) = ?$

where $x(t) = 10 + 5\cos 10t$

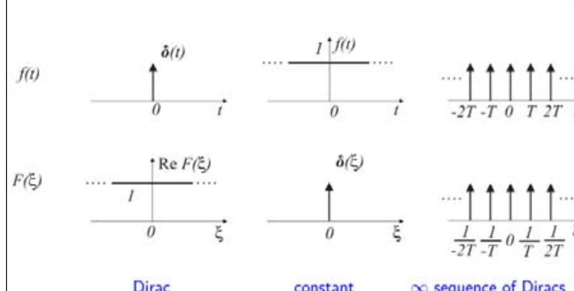
$H(\omega) = \frac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}} = \frac{1}{1+j\underbrace{\omega RC}_{10(1)(0.1)}} = \frac{1}{1+j1}$

$|H(\omega=10)| = \frac{1}{\sqrt{2}}$, $\angle H(\omega=10) = -45^\circ$

$y(t) = 10 + \frac{5}{\sqrt{2}}\cos(10t - 45^\circ)$

ans

Basic Fourier Tx pairs (1)



Basic Fourier Tx pairs (2)

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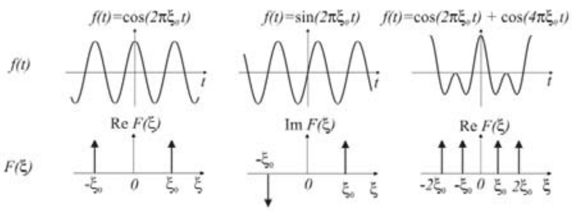

Basic Fourier Tx pairs (3)

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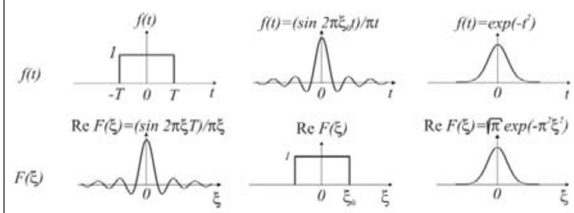

Uncertainty principle

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Position (x) & momentum (p) of a particle cannot be accurately measured at the same time:

$(\Delta x)(\Delta p) \geq h/4\pi$ h = Planck constant $= 6.626 \times 10^{-34}$ [J·s]

- All Fourier Tx pairs are constrained by the uncertainty principle. $J = kg\,m^2/s^2$
- The signal of short duration must have wide Fourier spectrum and vice versa.
- $\boxed{\text{(signal duration) (frequency bandwidth)} \geq \frac{1}{\pi}}$
- Observation: Gaussian e^{-t^2} modulated by a sinusoid (Gabor function) has the smallest duration-bandwidth product. e.g. $e^{-t^2}\cos\omega t$

- The principle is an instance of the general uncertainty principle introduced by Werner Heisenberg in quantum mechanics.

TABLE 5.2 Fourier Transform Pairs

Time Domain Signal	Fourier Transform
$f(t)$	$\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$
$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$	$F(\omega)$
$\delta(t)$	1
$A\delta(t-t_0)$	$Ae^{-j\omega t_0}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
1	$2\pi\delta(\omega)$
K	$2\pi K\delta(\omega)$
$\text{sgn}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$\text{rect}(t/T)$	$T\,\text{sinc}(\omega T/2)$
$\cos(\omega_0 t)u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)] + \frac{j\omega}{\omega_0^2-\omega^2}$
$\sin(\omega_0 t)u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)] + \frac{\omega_0}{\omega_0^2-\omega^2}$
$\text{rect}(t/T)\cos(\omega_0 t)$	$\frac{T}{2}\left[\text{sinc}\left(\frac{(\omega-\omega_0)T}{2}\right) + \text{sinc}\left(\frac{(\omega+\omega_0)T}{2}\right)\right]$
$\frac{\beta}{\pi}\,\text{sinc}(\beta t)$	$\text{rect}(\omega/2\beta)$
$\text{tri}(t/T)$	$T\,\text{sinc}^2(T\omega/2)$
$\text{sinc}^2(Tt/2)$	$\frac{2\pi}{T}\,\text{tri}(\omega/T)$
$e^{-at}u(t)$, $\text{Re}\{a\} > 0$	$\frac{1}{a+j\omega}$
$te^{-at}u(t)$, $\text{Re}\{a\} > 0$	$\left(\frac{1}{a+j\omega}\right)^2$
$t^{n-1}e^{-at}u(t)$, $\text{Re}\{a\} > 0$	$\frac{(n-1)!}{(a+j\omega)^n}$
$e^{-a\lvert t\rvert}$, $\text{Re}\{a\} > 0$	$\frac{2a}{a^2+\omega^2}$
$\sum_{n=-\infty}^{\infty} g(t-nT_0)$	$\sum_{n=-\infty}^{\infty} \omega_0 G(n\omega_0)\delta(\omega-n\omega_0)$, $\omega_0 = \frac{2\pi}{T_0}$
$\sum_{n=-\infty}^{\infty} g(t-nT_0) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$	$H(\omega) = \lvert H(\omega)\rvert\angle\phi(\omega) = \frac{\lvert V_2(\omega)\rvert\angle V_2}{\lvert V_1(\omega)\rvert\angle V_1} = \frac{\lvert V_2(\omega)\rvert}{\lvert V_1(\omega)\rvert}\angle V_2 - \angle V_1$
$\delta_T(t)$	$\sum_{k=-\infty}^{\infty} \omega_0\delta(\omega-k\omega_0)$

Fourier Transform of Periodic Functions

$f(t) = f(t+T_0)$, period T_0

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t} \qquad (A)$$

$$\Big\downarrow \mathcal{F}$$

$$\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t} e^{-j\omega t}\,dt = \sum_{k=-\infty}^{\infty} c_k \int_{-\infty}^{\infty} e^{-j(\omega-k\omega_0)t}\,dt$$

$$= 2\pi\sum_{k=-\infty}^{\infty} c_k\,\delta(\omega-k\omega_0)$$

rect(t/T_0)

(A) can be recast as $f(t) = g(t) * \delta_{T_0}(t)$

$g(t) = f(t)\,\text{rect}(t/T_0) \leftrightarrow G(\omega)$ generating function

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty}\delta(t-kT_0) = \sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t}$$

where $c_k = \frac{1}{T_0}\int_{T_0}\left(\sum_{n=-\infty}^{\infty}\delta(t-nT_0)\right)e^{-jk\omega_0 t}\,dt$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}}\Big(\delta(t) + \underbrace{\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\delta(t-nT_0)}_{\text{outside of } [-\frac{T_0}{2}, \frac{T_0}{2}]}\Big)e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{T_0}$$

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty}\left[\frac{1}{T_0}\right]e^{+jk\omega_0 t}$$

$$\Big\downarrow \mathcal{F} \qquad \boxed{e^{jk\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)}$$

$$\frac{1}{T_0}\sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega-k\omega_0) = \Delta_{T_0}(\omega)$$

$$\mathcal{F}\left[g(t) * \delta_{T_0}(t)\right] = G(\omega)\,\Delta_{T_0}(\omega)$$

$$= G(\omega)\left[\frac{2\pi}{T_0}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0)\right] = \omega_0\sum_{k=-\infty}^{\infty}G(k\omega_0)\,\delta(\omega-k\omega_0)$$

$2\pi/T_0 = \omega_0$