EE103 Lecture 23, Nov. 22, 2017

correction of Qz 6 solution

$$\mathcal{F}[f_1(t) f_2(t)] = \frac{1}{2\pi} F_1(\omega) * F_2(\omega)$$

missed

$$g_1(t) = 4\cos 100\pi t \quad \text{rect}(t \mid T = 10\times10^{-3})$$

$$\downarrow \mathcal{F}$$

$$G_1(\omega) = \frac{1}{2\pi}\Big[4\pi\big(\delta(\omega-100\pi) + \delta(\omega+100\pi)\big) * 10\times10^{-3}\,\text{sinc}(5\times10^{-3}\omega)\Big]$$

$$= 0.02\big[\text{sinc}\,(0.005(\omega-100\pi)) + \text{sinc}\,0.005(\omega+100\pi)\big]$$

TABLE 7.2 Laplace Transforms

$f(t), t \geq 0$	$F(s)$	ROC
1. $\delta(t)$	1	All s
2. $u(t)$	$\frac{1}{s}$	$Re(s) > 0$
3. t	$\frac{1}{s^2}$	$Re(s) > 0$
4. t^n	$\frac{n!}{s^{n+1}}$	$Re(s) > 0$
5. e^{-at}	$\frac{1}{s+a}$	$Re(s) > -a$
6. te^{-at}	$\frac{1}{(s+a)^2}$	$Re(s) > -a$
7. $t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$Re(s) > -a$
8. $\sin bt$	$\frac{b}{s^2+b^2}$	$Re(s) > 0$
9. $\cos bt$	$\frac{s}{s^2+b^2}$	$Re(s) > 0$
10. $e^{-at}\sin bt$	$\frac{b}{(s+a)^2+b^2}$	$Re(s) > -a$
11. $e^{-at}\cos bt$	$\frac{s+a}{(s+a)^2+b^2}$	$Re(s) > -a$
12. $t\sin bt$	$\frac{2bs}{(s^2+b^2)^2}$	$Re(s) > 0$
13. $t\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$	$Re(s) > 0$

$$\mathcal{L}[x(t)] \longrightarrow X(s) = \int_0^\infty x(t)e^{-st}\,dt$$

$\mathcal{L}[x(t-t_0)]$? $\quad x(t)$ delayed by t_0

$$\int_0^\infty x(t-t_0)\, e^{-st}\, dt$$

$$= \int_0^\infty x(t-t_0)\, e^{-s(t-t_0)}\, e^{-st_0}\, d(t-t_0)$$

$$\overset{t-t_0=\tau}{=} e^{-st_0}\int_0^\infty x(\tau)\, e^{-s\tau}\, d\tau = e^{-st_0}X(s)$$

$$\mathcal{L}\Big[\int_{-\infty}^t x(t)\,dt\Big] = ? \qquad x(t) \xrightarrow{\mathcal{L}} X(s)$$

$$\int_0^\infty \int_{-\infty}^t x(\tau)\,d\tau \;\, e^{-st}\,dt$$

$$= \int_0^\infty \Big(\int_{-\infty}^t x(\tau)\,d\tau\Big)\, d\Big(\frac{e^{-st}}{-s}\Big)$$

$$= \Big(\int_{-\infty}^t x(\tau)\,d\tau\Big)\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\,x(t)\,dt$$

$$= \frac{1}{s}\int_0^\infty x(t)\,e^{-st}\,dt = \frac{1}{s}X(s)$$

Properties of $\mathcal{L}$

$$\mathcal{L}(a x_1(t) + b x_2(t)) = aX_1(s) + bX_2(s)$$

LHS $\int_0^\infty (a x_1(t) + b x_2(t))\, e^{-st}\, dt$

$$= a\int_0^\infty x_1(t)\, e^{-st} + b\int_0^\infty x_2(t)\, e^{-st}\,dt$$

$$= aX_1(s) + bX_2(s)$$

TABLE 7.3 Laplace Transform Properties

Name	Property
1. Linearity, (7.10)	$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1F_1(s) + a_2F_2(s)$
2. Derivative, (7.15)	$\mathcal{L}\Big[\frac{df(t)}{dt}\Big] = sF(s) - f(0^+)$
3. nth-order derivative, (7.29)	$\mathcal{L}\Big[\frac{d^nf(t)}{dt^n}\Big] = s^nF(s) - s^{n-1}f(0^+)$ $- \cdots - sf^{(n-2)}(0^+) - f^{(n-1)}(0^+)$
4. Integral, (7.31)	$\mathcal{L}\Big[\int_0^t f(\tau)d\tau\Big] = \frac{F(s)}{s}$
5. Real shifting, (7.22)	$\mathcal{L}[f(t-t_0)u(t-t_0)] = e^{-t_0s}F(s)$
6. Complex shifting, (7.20)	$\mathcal{L}[e^{-at}f(t)] = F(s+a)$
7. Initial value, (7.36)	$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$
8. Final value, (7.39)	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$
9. Multiplication by t, (7.34)	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
10. Time transformation, (7.42) ($a > 0; b \geq 0$)	$\mathcal{L}[f(at-b)u(at-b)] = \frac{e^{-sb/a}}{a}F\Big(\frac{s}{a}\Big)$
11. Convolution	$\mathcal{L}^{-1}[F_1(s)F_2(s)] = \int_0^t f_1(t-\tau)f_2(\tau)\,d\tau$ $= \int_0^t f_1(\tau)f_2(t-\tau)d\tau$
12. Time periodicity $[f(t) = f(t+T)], t \geq 0$	$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}}F_1(s)$, where $F_1(s) = \int_0^T f(t)e^{-st}\,dt$

$$\mathcal{L}[t f(t)] =$$

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^\infty (-t) f(t) e^{-st} dt$$

$$= -\mathcal{L}[t f(t)]$$

example $f(t) = u(t) \xrightarrow{\mathcal{L}} \frac{1}{s}$

$$\frac{d}{ds}\left(\frac{1}{s}\right) = -\frac{1}{s^2} = -\underbrace{\mathcal{L}[t]}_{=\frac{1}{s^2}}$$

$$\mathcal{L}[f_1(t)*f_2(t)] = \mathcal{L}\left[\int_{-\infty}^{\infty} f_1(t-\tau)f_2(\tau)d\tau\right]$$

$$= \int_0^\infty \int_{-\infty}^\infty f_1(t-\tau) f_2(\tau)\, d\tau\, e^{-st} dt$$

$$= \int_0^\infty \int_{-\infty}^\infty f_1(t-\tau) e^{-s(t-\tau)} e^{-s\tau} f_2(\tau)\, d\tau\, d(t-\tau)$$

$$= \int_0^\infty f_1(t') e^{-st'}dt' \int_0^\infty f_2(\tau)e^{-s\tau}d\tau$$

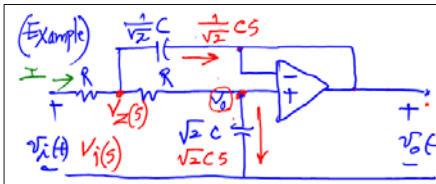
$0 < t < \infty$

$-\infty < \tau < \infty$

$0 < t-\tau < \infty$

t'

$$= F_1(s)\, F_2(s)$$



Find $H(s) = \frac{V_o(s)}{V_i(s)}$

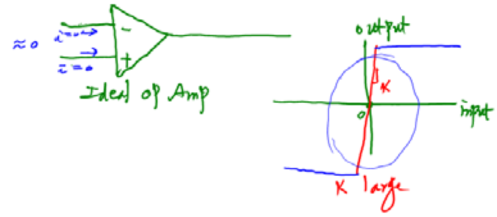
$$V_i(s) = RI + V_Z(s)$$

$$V_Z(s) = R(\sqrt{2}Cs\,V_o(s)) + V_o(s)$$

$$I = [V_Z(s) - V_o(s)]\frac{1}{\sqrt{2}}Cs + \sqrt{2}Cs\,V_o(s)$$

$$V_i(s) = R\left[(R\sqrt{2}Cs\,V_o + V_o - V_o)\frac{1}{\sqrt{2}}Cs + \sqrt{2}Cs\,V_o\right] + R\sqrt{2}Cs\,V_o + V_o$$

$$= R[R(Cs)^2 V_o + \sqrt{2}Cs\,V_o] + R\sqrt{2}Cs\,V_o + V_o$$



$$\frac{V_o(s)}{V_i(s)} = \frac{1}{R^2C^2s^2 + 2\sqrt{2}RCs + 1}$$

$$= \frac{1}{R^2C^2}\,\frac{1}{s^2 + 2\frac{\sqrt{2}}{RC}s + \frac{1}{R^2C^2}}$$

$$= \frac{1}{R^2C^2}\,\frac{1}{\left(s + \frac{\sqrt{2}}{RC}\right)^2 - \frac{1}{R^2C^2}}$$

$$= \frac{1}{R^2C^2}\,\frac{1}{\left(s + \frac{\sqrt{2}}{RC} + \frac{1}{RC}\right)\left(s + \frac{\sqrt{2}}{RC} - \frac{1}{RC}\right)}$$

$$= \frac{1}{R^2C^2}\,\frac{1}{\left(s + \frac{\sqrt{2}+1}{RC}\right)\left(s + \frac{\sqrt{2}-1}{RC}\right)}$$

$$\frac{1}{(s+p_1)(s+p_2)} = \frac{A}{s+p_1} + \frac{B}{s+p_2}$$

$$A e^{-p_1 t} + B e^{-p_2 t}\; u(t)$$

$$\cancel{(s+p_1)}\frac{1}{\cancel{(s+p_1)}(s+p_2)}\Bigg|_{s=-p_1} = A + \frac{B\cancel{(s+p_1)}}{s+p_2}\Bigg|_{s=-p_1}$$

0

$$\boxed{\frac{1}{p_2-p_1} = A}$$

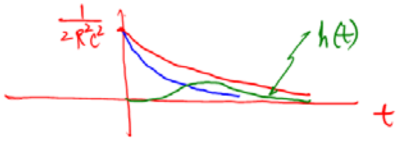
$$\cancel{(s+p_2)}\frac{1}{(s+p_1)\cancel{(s+p_2)}}\Bigg|_{s=-p_2} = \frac{A(s+p_2)}{s+p_1}\Bigg|_{s=-p_2} + B$$

$$\boxed{\frac{1}{p_1-p_2} = B}$$

$$= \frac{1}{2R^2C^2}\left(\frac{-1}{s+\frac{\sqrt{2}+1}{RC}} + \frac{1}{s+\frac{\sqrt{2}-1}{RC}}\right)$$

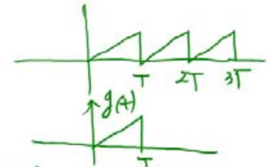
$$\downarrow \mathcal{L}^{-1}$$

$$h(t) = \frac{1}{2R^2C^2}\left(e^{-\frac{(\sqrt{2}-1)}{RC}t} - e^{-\frac{(\sqrt{2}+1)}{RC}t}\right)u(t)$$



Time Periodicity

$$f(t) = f(t+T)$$



$$f(t) = g(t) * \delta_T(t)$$

$$k \geq 0$$

$$\downarrow \mathcal{L}$$

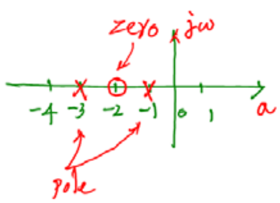
$$F(s) = G(s)\left[1 + e^{-sT} + e^{-2sT} + \cdots\right]$$

$$= G(s)\,\frac{1}{1-e^{-sT}}$$

Example 7.13 (page 368)

$$H(s) = \frac{4s+8}{2s^2+8s+6} = \frac{2(s+2)}{s^2+4s+3}$$

$$= \frac{2(s+4)}{(s+3)(s+1)} = \left[\frac{-1}{s+3} + \frac{+1}{s+1}\right]$$



$$\downarrow \mathcal{L}^{-1}$$

$$h(t) = \left(e^{-t} - e^{-3t}\right)u(t)$$