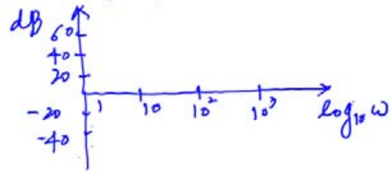
EE 103 Lect #16 Nov. 3, 2017

HW 5 posted

QZ 5 Nov. 6 (M)

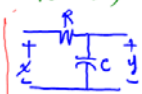
Bode plot



Textbook Example 5.21 (page 248-250)

$x(t) \rightarrow \boxed{H} \rightarrow y(t)$

$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$



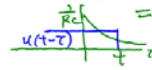
$H(s) = \frac{1/Cs}{R + 1/Cs} = \frac{1}{RCs+1} = \frac{1}{RC}\frac{1}{s+\frac{1}{RC}}$

$\downarrow$

$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$

$x(t) = V\,u(t)$

$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} \frac{1}{RC} e^{-\tau/RC} u(\tau) \times V u(t-\tau)\,d\tau$



$= V\int_0^t \frac{1}{RC} e^{-\tau/RC}\,d\tau$

$= V(1 - e^{-\frac{1}{RC}t})\,u(t)$

$X(\omega) = \mathcal{F}[x(t)] = V\mathcal{F}[u(t)] = V\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$

$H(\omega) = \int_{-\infty}^{\infty} \underbrace{\frac{1}{RC} e^{-t/RC} u(t)}\, e^{-j\omega t}\,dt$

$= \frac{1}{RC}\int_0^{\infty} e^{-\left(\frac{1}{RC} + j\omega\right)t}\,dt = \frac{1}{RC}\left.\frac{e^{-\left(\frac{1}{RC}+j\omega\right)t}}{-\left(\frac{1}{RC}+j\omega\right)}\right|_0^{\infty}$

$= \frac{1}{RC}\left(0 + \frac{1}{\frac{1}{RC}+j\omega}\right) = \frac{1}{1+j\omega RC}$

$Y(\omega) = H(\omega)X(\omega) = \frac{1}{1+j\omega RC}\, V\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$

$= V\left[\frac{1}{1+j\omega RC}\frac{1}{j\omega} + \underbrace{\frac{1}{1+j\omega RC}\pi\delta(\omega)}_{=\pi\delta(\omega)}\right]$

$= V\left[\frac{-RC}{1+j\omega RC} + \frac{1}{j\omega} + \pi\delta(\omega)\right]$

$y(t) = V\left[-u(t)e^{-t/RC} + u(t)\right] = V(1 - e^{-\frac{t}{RC}})u(t)$

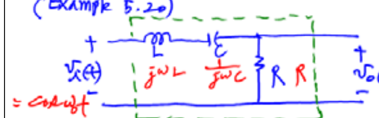
Bode plot of $H(\omega)$

section 5.4 pp. 245-248 (Textbook)

$X(\omega) \rightarrow \boxed{H(\omega)} \rightarrow Y(\omega)$

$H(\omega) = |H(\omega)| \angle H(\omega) = |H(\omega)|\, e^{j\angle H(\omega)}$

(Example 5.20)



$= A\cos(\omega_0 t + \phi)$

$H(\omega) = \frac{R}{j\omega L + \frac{1}{j\omega C} + R} = \frac{(j\omega C)R}{1 + (j\omega L)(j\omega C) + j\omega RC}$

$= \frac{j\omega RC}{(1 - \omega^2 LC + j\omega RC)}$

In this circuit, at $\omega = 1/\sqrt{LC}$ $j\omega L + \frac{1}{j\omega C} \simeq 0$ (resonance) and $H(\omega) = 1$

$H(\omega) = |H(\omega)| \angle H(j\omega) = |H(\omega)|\, e^{j\angle H(\omega)}$

$|H(\omega)| = \frac{|j\omega RC|}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}} = \frac{\omega RC}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}}$

$\boxed{\text{(Case 1)}\ L = 1\ mH,\ C = 1\ \mu F,\ R = 1\ M\Omega}$

then $|H(\omega)| = \frac{\omega(10^6)(10^{-6})}{\sqrt{(1-\omega^2(10^{-3})(10^{-6}))^2 + (\omega 10^6 \cdot 10^{-6})^2}}$

$= \frac{\omega}{\sqrt{(1-\omega^2 10^{-9})^2 + \omega^2}}$

- $\omega = 0$ $|H(\omega)| = 0$; • $\omega = 1$, $|H(1)| = \frac{1}{\sqrt{(1-10^{-9})^2+1}} \doteq 1$
- $\omega = 10$ $|H(10)| = \frac{10}{\sqrt{(1-10^2 \cdot 10^{-9})^2 + 10^2}} = 1$

- $\omega = 10^2$ $|H(10^2)| = \frac{10^2}{\sqrt{(1-(10^2)^2 10^{-9})^2 + (10^2)^2}} = 1$

  ≈ 1

- $\omega = 10^3$ $|H(10^3)| = \frac{10^3}{\sqrt{(1-(10^3)^2 10^{-9})^2 + (10^3)^2}} = 1$

  $\approx \frac{10^3}{10^3\sqrt{2}} = \frac{1}{\sqrt{2}}$

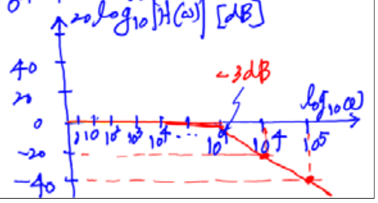
- $\omega = 10^4$ $|H(10^4)| = \frac{10^4}{\sqrt{(1-(10^4)^2 10^{-9})^2 + (10^4)^2}} = \frac{10^4}{\sqrt{10^8 + 0.81}}$

  $\doteq 1$

  ⋮

- $\omega = 10^9$ $|H(10^9)| = \frac{10^9}{\sqrt{(1-(10^9)^2 10^{-9})^2 + (10^9)^2}} = \frac{1}{\sqrt{2}}$

ω	$\lvert H(\omega)\rvert$	$20\log_{10}\lvert H(\omega)\rvert$
0	0	$-\infty$
$10^0 = 1$	1	0
10^2	1	0
10^3	1	
10^4	1	
10^5	1	
10^9	$\frac{1}{\sqrt{2}}$	

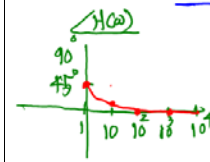


Phase angle vs. ω plot

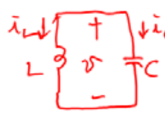
$$\angle H(\omega) = \angle \frac{j\omega RC}{(1-\omega^2 LC) + j\omega RC} = 90^\circ - \tan^{-1}\frac{\omega}{1-\omega^2(10^{-9})}$$

$R = 10^6\,\Omega$
$C = 1\,\mu F$
$L = 1\,mH$

ω	$\angle H(\omega)$
0	90°
1	45°
10	$90^\circ - \tan^{-1}\left(\frac{10}{1-10^{-7}}\right) = 5.71^\circ$
10^2	$90^\circ - \tan^{-1}\frac{10^2}{(1-10^{-5})} \approx 0$
10^3	$90^\circ - \tan^{-1}\frac{10^3}{(1-10^{-3})} \approx 0$



Example LC tank circuit



$v(t) = V\cos\omega t = V\cos\frac{1}{\sqrt{LC}}t$
$v(0) = V$ initial voltage across C

$i_C = C\frac{dv}{dt}$ $\quad v = L\frac{di_L}{dt} = L\frac{d}{dt}(-i_C) = L\frac{d}{dt}\left[-C\frac{dv}{dt}\right] = -LC\frac{d^2v}{dt^2}$

$\boxed{LC\frac{d^2v}{dt^2} + v = 0}$ If $v = V\cos\omega t$

$LC(-\omega^2 V\cos\omega t) + V\cos\omega t = 0 \Rightarrow LC\omega^2 = 1$

$\omega = \frac{1}{\sqrt{LC}}$ (resonant freq) $\quad \boxed{v(t) = V\cos\frac{1}{\sqrt{LC}}t}$

Parseval's Theorem

$$\int_{-\infty}^{\infty} \lvert f(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert F(\omega)\rvert^2 d\omega$$

$f(t) = \frac{1}{2\pi}\int F(\omega)e^{+j\omega t}d\omega$

Proof $\int_{-\infty}^{\infty}\lvert f(t)\rvert^2 dt = \int_{-\infty}^{\infty} f(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{+j\omega t}d\omega\right)dt$

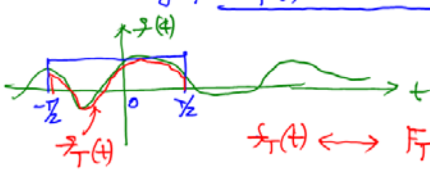
$= \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)\underbrace{\int_{-\infty}^{\infty}f(t)e^{+j\omega t}dt}_{F(-\omega)=F^*(\omega)}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert F(\omega)\rvert^2 d\omega$ ⊠

For power signal $f(t)$,

$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\lvert f(t)\rvert^2 dt < \infty$$

(examples) $u(t)$, $\mathrm{sgn}(t)$, periodic signals
But their energy is infinite!

Truncated signal $f_T(t) = f(t)\,\mathrm{rect}(t/T)$



$f_T(t) \longleftrightarrow F_T(\omega)$

$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\lvert f_T(t)\rvert^2 dt$$

$$\int_{-\infty}^{\infty}\lvert f_T(t)\rvert^2 dt \underset{\text{Parseval's Theorem}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert F_T(\omega)\rvert^2 d\omega$$

$$P = \lim_{T\to\infty}\frac{1}{T}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert F_T(\omega)\rvert^2 d\omega\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\lim_{T\to\infty}\frac{1}{T}\lvert F_T(\omega)\rvert^2}_{\mathcal{P}_f(\omega)\text{ power spectral density}}d\omega$$

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{P}_f(\omega)\,d\omega$$

from the Fourier series of $f(t)$

$$f(t)=\sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t}$$

$$F(\omega)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}F(k\omega_0)\,\delta(\omega-k\omega_0)$$

Power $P=\sum_{k=-\infty}^{\infty}|c_k|^2 = c_0^2+2\sum_{k=1}^{\infty}|c_k|^2$

$$=\frac{1}{4\pi^2}F(0)+\frac{1}{2\pi^2}\sum_{k=1}^{\infty}|F(k\omega_0)|^2$$

$$\frac{1}{2\pi}F(k\omega_0)=c_k$$

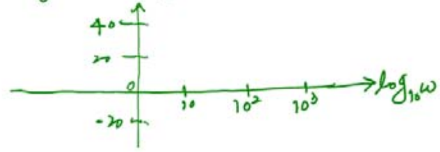
$x(t)\rightarrow \boxed{H} \rightarrow y(t)$

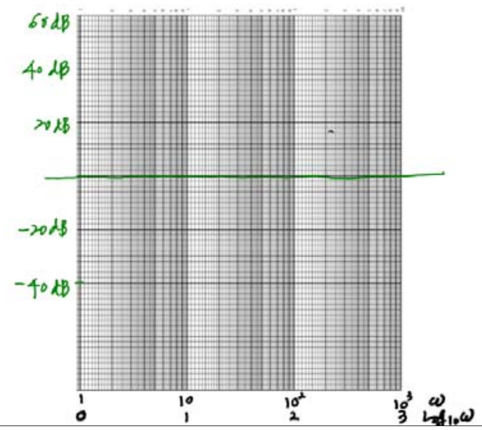
$$Y(\omega)=H(\omega)X(\omega)$$

$$|Y(\omega)|^2=|H(\omega)|^2|X(\omega)|^2$$

$$10\log_{10}|Y(\omega)|^2 = \underbrace{20\log_{10}|H(\omega)|} + 10\log_{10}|X(\omega)|^2$$

$20\log_{10}|H(\omega)|$ [dB]





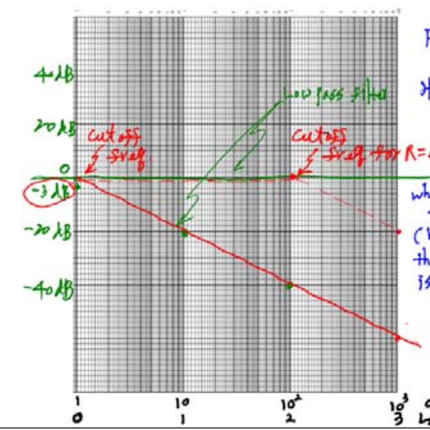
10 MΩ, $\cos\omega t$, 0.1 μF

$$H(\omega)=\frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C}+R}=\frac{1}{1+j\omega RC}$$

For $R=1$, $H(\omega)=\frac{1}{1+j\omega(10^7\cdot 10^{-7})}=\frac{1}{1+j\omega}$

$$|H(\omega)|=\frac{1}{\sqrt{1+\omega^2}}$$

- $\omega=1$ $\quad \frac{1}{\sqrt{2}}$ $\quad 20\log 2^{-\frac{1}{2}}=-10\log_{10}2=-3$ dB
- 10 $\quad \frac{1}{10}$ $\quad 20\log_{10}10^{-1}=-20$ dB
- 10^2 $\quad \frac{1}{10^2}$ $\quad 20\log_{10}10^{-2}=-40$ dB



For $R=0.1$ MΩ, $C=0.1$ μF,

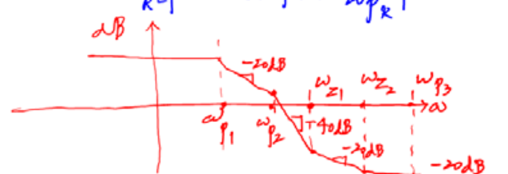
$$H(\omega)=\frac{1}{1+j\omega/100}$$

when R is reduced to 0.1 MΩ (by a factor of $\frac{1}{100}$) then the 3dB point is increased by 100. (- - - -)

In general

$$H(\omega)=K\frac{\left(1+\frac{\omega}{\omega_{z_1}}\right)\left(1+\frac{\omega}{\omega_{z_2}}\right)\cdots\left(1+\frac{\omega}{\omega_{z_M}}\right)}{\left(1+\frac{\omega}{\omega_{p_1}}\right)\left(1+\frac{\omega}{\omega_{p_2}}\right)\cdots\left(1+\frac{\omega}{\omega_{p_N}}\right)}$$

$$20\log_{10}|H(\omega)|=20\log_{10}|K|+\sum_{j=1}^{M}20\log_{10}\left|1+\frac{\omega}{\omega_{z_j}}\right|-\sum_{k=1}^{N}20\log_{10}\left|1+\frac{\omega}{\omega_{p_k}}\right|$$



Example

$$H(\omega) = 100 \, \frac{\left(1+\frac{\omega}{100}\right)}{\left(1+\frac{\omega}{10}\right)\left(1+\frac{\omega}{1000}\right)}$$

$$20\log_{10}|H(\omega)| = \underbrace{20\log_{10}100}_{=40} + 20\log_{10}\left|1+\frac{\omega}{100}\right| - 20\log_{10}\left|1+\frac{\omega}{10}\right| - 20\log_{10}\left|1+\frac{\omega}{1000}\right|$$