EE 103 lect #15 Oct. 30, 2017

HW #5 posted
QUIZ 4 Today ; Midterm on 11/1

Fourier transform of $x(t) = e^{-|t|}$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-|t|} e^{-j\omega t} dt = \int_{-\infty}^{0} e^{+t} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-t} e^{-j\omega t} dt$$

$$= \frac{e^{(1-j\omega)t}}{1-j\omega}\Big|_{-\infty}^{0} + \frac{e^{-(1+j\omega)t}}{-(1+j\omega)}\Big|_{0}^{\infty}$$

$$= \frac{1}{1-j\omega} - 0 + 0 + \frac{1}{1+j\omega} = \frac{2}{1+\omega^2}$$

$$\mathcal{F}\left[\frac{d}{dt} e^{-|t|}\right] = (j\omega)\, \mathcal{F}\left[e^{-|t|}\right] = j\omega \frac{2}{1+\omega^2}$$

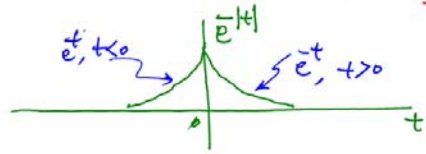
Table 5-1
$\frac{d^n}{dt^n} f(t) \leftrightarrow (j\omega)^n F(\omega)$

$$\mathcal{F}\int_{-\infty}^{\infty}\left(\frac{d}{dt} e^{-|t|}\right) e^{-j\omega t} dt = \int_{-\infty}^{0}\left(\frac{d}{dt} e^{t}\right) e^{-j\omega t} dt + \int_{0}^{\infty}\left(\frac{d}{dt} e^{-t}\right) e^{-j\omega t} dt = \int_{-\infty}^{0} e^{+t} e^{-j\omega t} dt - \int_{0}^{\infty} e^{-t} e^{-j\omega t} dt$$

$$= \frac{e^{(1-j\omega)t}}{(1-j\omega)}\Big|_{-\infty}^{0} - \frac{e^{-(1+j\omega)t}}{-(1+j\omega)}\Big|_{0}^{\infty}$$

$$= \left(\frac{1}{1-j\omega} - 0\right) - \left(0 - \frac{1}{-(1+j\omega)}\right)$$

$$= \frac{1}{1-j\omega} - \frac{1}{1+j\omega} = \frac{1+j\omega-1+j\omega}{1+\omega^2} = j\omega \frac{2}{1+\omega^2}$$



$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t} \quad \text{(periodic case)}$$

$$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} c_k (jk\omega_0) e^{+jk\omega_0 t}$$

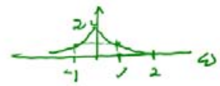
As $T_0 \to \infty$ $(\omega_0 \to 0)$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega t}, \quad \frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} c_k (j\omega) e^{+j\omega t} = (j\omega)\sum_{k=-\infty}^{\infty} c_k e^{+j\omega t}$$

$\downarrow \mathcal{F}$ $\quad\quad \downarrow \mathcal{F}$

$X(\omega) \quad\quad j\omega X(\omega)$

For $x(t) = e^{-|t|} \leftrightarrow X(\omega) = \frac{2}{1+\omega^2}$



$$\hat{x}(t) = e^{-|t|} \cos\omega_0 t \xrightarrow{\mathcal{F}} \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right] * \frac{2}{1+\omega^2}$$

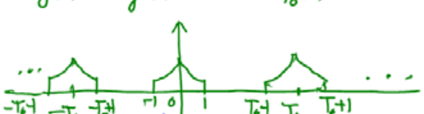
$$= 2\pi\left[\frac{1}{1+(\omega-\omega_0)^2} + \frac{1}{1+(\omega+\omega_0)^2}\right]$$

$$\cos\omega_0 t = \left[\frac{e^{+j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]$$

$\mathcal{F} \downarrow$

$$\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

$$y(t) = g(t) * \sum \delta_{T_0}(t)$$



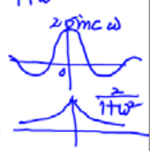
$$g(t) = e^{-|t|}\, \text{rect}(t/T)$$

$$Y(\omega) = G(\omega)\, \Delta_{T_0}(\omega)$$

$$\frac{1}{T_0}\sum 2\pi\delta(\omega - k\omega_0) = \omega_0 \sum \delta(\omega - k\omega_0)$$

$$= \omega_0 \sum G(k\omega_0)\, \delta(\omega - k\omega_0)$$

$$G(\omega) = \mathcal{F}\left[e^{-|t|}\right] * \mathcal{F}\left[\text{rect}(t/2)\right] = \frac{2}{1+\omega^2} * 2\,\text{sinc}\,\omega$$



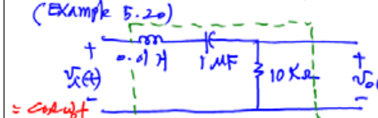
Bode plot of $H(\omega)$

section 5.4 pp. 245-248 (Textbook)

$X(\omega) \rightarrow \boxed{H(\omega)} \rightarrow Y(\omega)$

$$H(\omega) = |H(\omega)| \angle H(\omega) = |H(\omega)|\, e^{j\angle H(\omega)}$$

(Example 5.20)



$= \cos\omega_0 t$ $\qquad = A\cos(\omega_0 t + \phi)$

$$H(\omega) = \frac{10^4}{j0.01\,\omega + \frac{1}{j\omega(10^{-6})} + 10^4} = \frac{j\omega\, 10^{-2}}{1 - 10^{-8}\omega^2 + j\omega 10^{-2}}$$

$$|H(\omega)| = \frac{10^{-2}\omega}{\sqrt{(1-10^{-8}\omega^2)^2 + (10^{-2}\omega)^2}} \qquad \angle H(\omega) = 90^\circ - \tan^{-1}\frac{10^{-2}\omega}{1-10^{-8}\omega^2}$$

At $\omega=0$ $H(\omega=0) = \frac{10^4}{j0.01(0) + \frac{1}{j(0)10^{-6}} + 10^4} = 0$

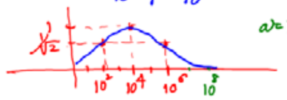
$\omega = 10^4$, $|H(\omega)| = \frac{10^{-2}(10^4)}{\sqrt{(1-10^{-8}(10^4)^2)^2 + (10^{-2}10^4)^2}}$

$= \frac{10^{-2}(10^4)}{\sqrt{0 + 10^4}} = 1$

$\omega = 10^6$, $H(\omega) = \frac{10^{-2}(10^6)}{\sqrt{(1-10^{-8}(10^6)^2)^2 + (10^{-2}10^6)^2}}$

$= \frac{10^4}{\sqrt{10^8 + 10^8}} = \frac{1}{\sqrt{2}}$

$\omega = 10^2$: $\frac{10^{-2}(10^2)}{\sqrt{[1-10^{-8}(10^2)^2]^2 + [10^{-2}(10^2)]^2}} = \frac{1}{\sqrt{2}}$



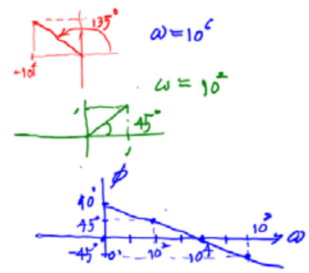
$\omega = 10^8$: $\frac{10^{-2}(10^8)}{\sqrt{[(1-10^{-8}(10^8)^2]^2 + [10^{-2}(10^8)]^2}} = 10^{-2}$

$\angle H(\omega)$, At $\omega = 0$ $\angle = 90^\circ - \tan^{-1}\frac{10^{-2}\omega}{1-10^{-8}\omega^2} = 90^\circ$

$\omega = 10^4$: $90^\circ - \tan^{-1}\frac{10^2}{1-10^{-8}(10^4)^2} = 0$

$\omega = 10^6$: $90^\circ - \tan^{-1}\frac{10^{-2}(10^6)}{1-10^{-8}(10^6)^2} = -45^\circ$

$\omega = 10^2$: $90^\circ - \tan^{-1}\frac{10^{-2}(10^2)}{1-10^{-8}(10^2)^2} = 45^\circ$



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Fourier transform of $x(t) = e^{-|t|}$

$X(\omega) = \int_{-\infty}^{\infty} e^{-|t|} e^{-j\omega t}\,dt = \int_{-\infty}^{0} e^{+t} e^{-j\omega t}\,dt + \int_0^\infty e^{-t} e^{-j\omega t}\,dt$

$= \frac{e^{(1-j\omega)t}}{1-j\omega}\Big|_{-\infty}^{0} + \frac{e^{-(1+j\omega)t}}{-(1+j\omega)}\Big|_0^\infty$

$= \frac{1}{1-j\omega} - 0 + 0 + \frac{1}{1+j\omega} = \frac{2}{1+\omega^2}$

$\mathcal{F}\left[\frac{d}{dt} e^{-|t|}\right] = (j\omega)\,\mathcal{F}\left[e^{-|t|}\right] = j\omega \frac{2}{1+\omega^2}$

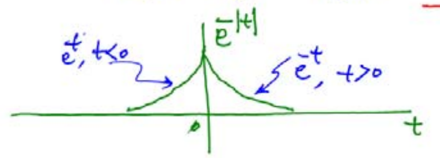
Table 5-1: $\frac{d^n}{dt^n} f(t) \leftrightarrow (j\omega)^n F(\omega)$

$\mathcal{F}: \int_{-\infty}^{\infty}\left(\frac{d}{dt} e^{-|t|}\right) e^{-j\omega t}\,dt = \int_{-\infty}^{0}\left(\frac{d}{dt} e^{t}\right) e^{-j\omega t}\,dt + \int_0^\infty \left(\frac{d}{dt} e^{-t}\right) e^{-j\omega t}\,dt = \int_{-\infty}^{0} e^{t} e^{-j\omega t}\,dt - \int_0^\infty e^{-t} e^{-j\omega t}\,dt$

$= \frac{e^{(1-j\omega)t}}{(1-j\omega)}\Big|_{-\infty}^{0} - \frac{e^{-(1+j\omega)t}}{-(1+j\omega)}\Big|_0^\infty$

$= \left(\frac{1}{1-j\omega} - 0\right) - \left(0 - \frac{1}{-(1+j\omega)}\right)$

$= \frac{1}{1-j\omega} - \frac{1}{1+j\omega} = \frac{1+j\omega-1+j\omega}{1+\omega^2} = j\omega\frac{2}{1+\omega^2}$



$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t}$ (periodic case)

$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} c_k (jk\omega_0) e^{+j\omega_0 t}$

As $T_0 \to \infty$ $(\omega_0 \to 0)$

$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega t}$, $\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} c_k (j\omega) e^{+j\omega t} = (j\omega)\sum_{k=-\infty}^{\infty} c_k e^{+j\omega t}$

$\downarrow \mathcal{F}$ $\quad \downarrow \mathcal{F}$

$X(\omega)$ $\quad j\omega X(\omega)$

For $x(t) = e^{-|t|} \leftrightarrow X(\omega) = \frac{2}{1+\omega^2}$

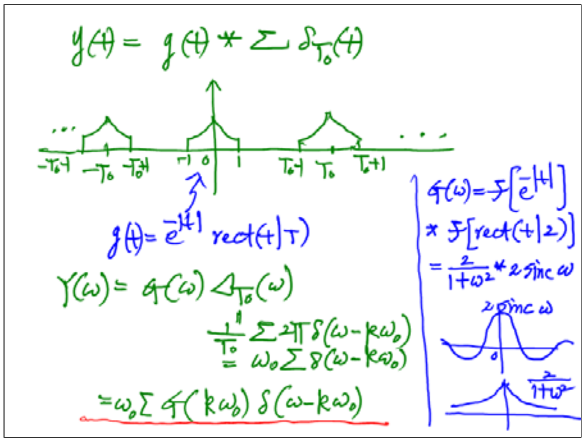
$\hat{x}(t) = e^{-|t|}\cos\omega_0 t \xrightarrow{\mathcal{F}} \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right] * \frac{2}{1+\omega^2}$

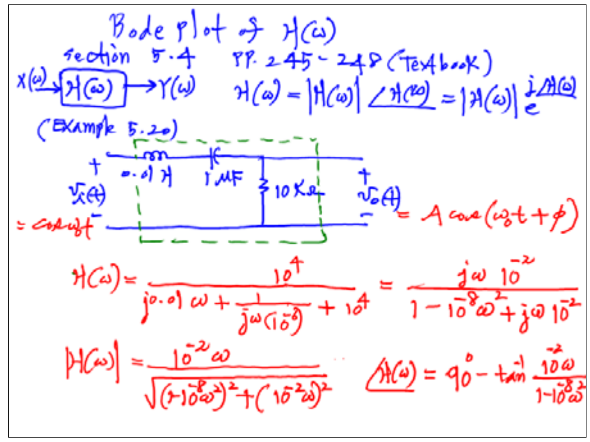
$= 2\pi\left[\frac{1}{1+(\omega-\omega_0)^2} + \frac{1}{1+(\omega+\omega_0)^2}\right]$

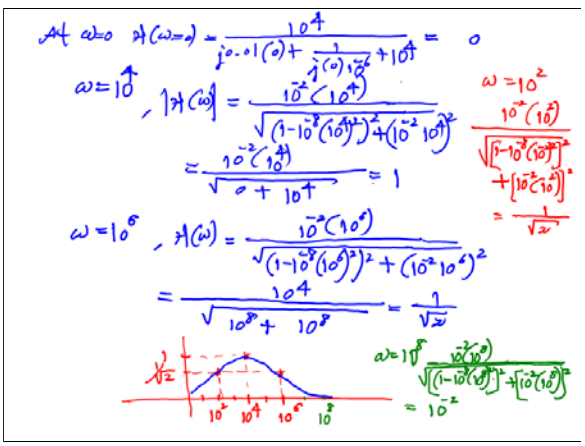
$\cos\omega_0 t = \left[\frac{e^{+j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]$

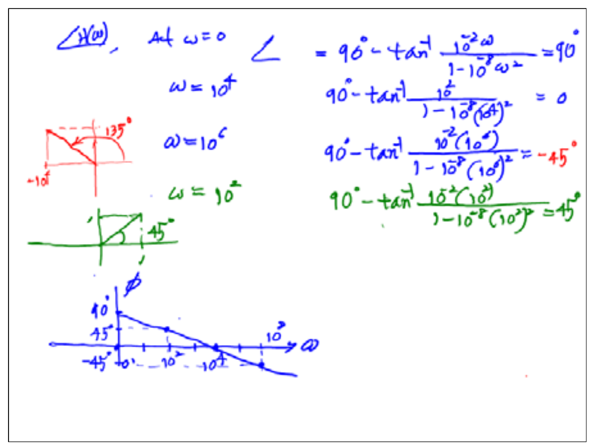
$\mathcal{F}\downarrow$

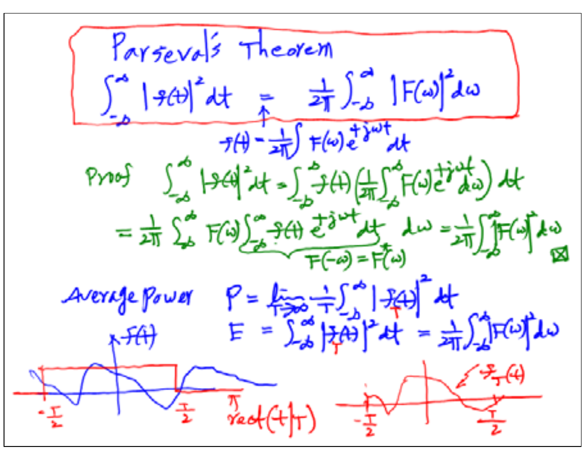
$\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$

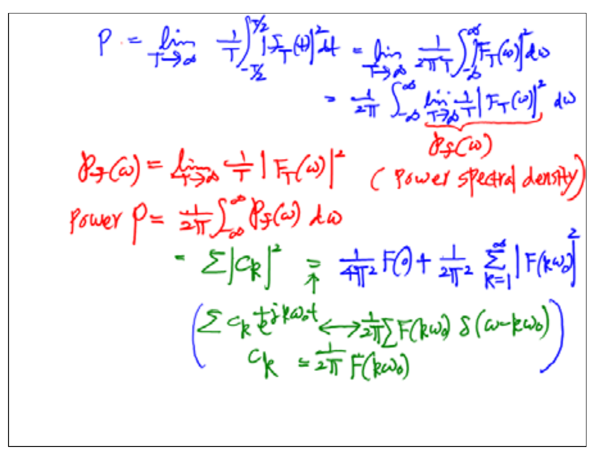

y(t) = g(t) * Σ δ_T0(t)
g(t) = e^-|t| rect(t|T)
Y(ω) = G(ω) Δ_T0(ω)
= 1/T0 Σ 2π δ(ω - kω0)
= ω0 Σ δ(ω - kω0)
= ω0 Σ G(kω0) δ(ω - kω0)
G(ω) = F[e^-|t|]
* F[rect(t|2)]
= 2/(1+ω²) * 2 sinc ω
2 sinc ω
2/(1+ω²)

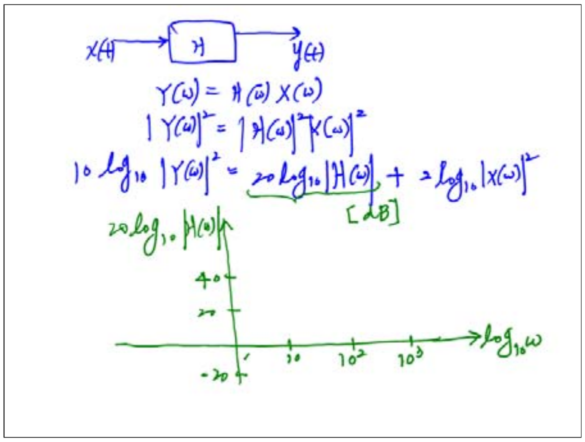

Bode plot of H(ω)
section 5.4 pp. 245 - 248 (Textbook)
x(ω) → H(ω) → Y(ω)
H(ω) = |H(ω)| ∠H(ω) = |H(ω)| e^{j∠H(ω)}
(Example 5.20)
0.01 H
1 μF
10 KΩ
vi(t)
vo(t)
= cos ω0t
= A cos(ω0t + φ)
H(ω) = 10^4 / (j0.01 ω + 1/(jω(10^-6)) + 10^4) = jω 10^-2 / (1 - 10^-8 ω² + jω 10^-2)
|H(ω)| = 10^-2 ω / √((1-10^-8 ω²)² + (10^-2 ω)²)
∠H(ω) = 90° - tan^-1 (10^-2 ω / (1 - 10^-8 ω²))


At ω=0 H(ω=0) = 10^4 / (j0.01(0) + 1/(j(0)10^-6) + 10^4) = 0
ω = 10^4, |H(ω)| = 10^-2(10^4) / √((1-10^-8(10^4)²)² + (10^-2 10^4)²)
= 10^-2(10^4) / √(0 + 10^4) = 1
ω = 10^6, H(ω) = 10^-2(10^6) / √((1-10^-8(10^6)²)² + (10^-2 10^6)²)
= 10^4 / √(10^8 + 10^8) = 1/√2
ω = 10^2
10^-2(10^2) / √([1-10^-8(10^2)²]² + [10^-2(10^2)]²) = 1/√2
ω = 10^8
10^-2(10^8) / √([(1-10^-8(10^8)²]² + [10^-2(10^8)]²) = 10^-2
1/√2
10^2 10^4 10^6 10^8

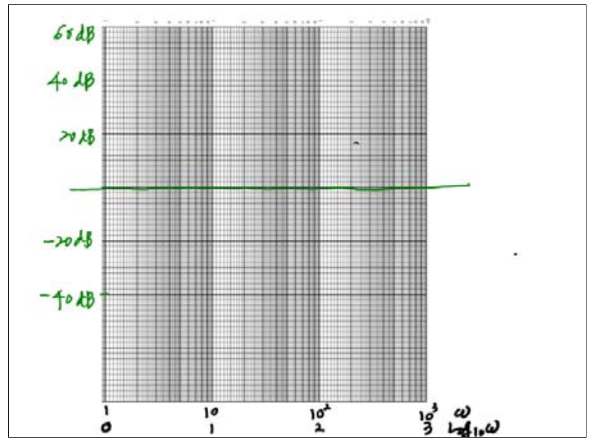

∠H(ω), At ω = 0 ∠ = 90° - tan^-1 (10^-2 ω / (1 - 10^-8 ω²)) = 90°
ω = 10^4 90° - tan^-1 (10^2 / (1 - 10^-8(10^4)²)) = 0
ω = 10^6 90° - tan^-1 (10^-2(10^6) / (1 - 10^-8(10^6)²)) = -45°
ω = 10^2 90° - tan^-1 (10^-2(10^2) / (1 - 10^-8(10^2)²)) = 45°
135°
45°
φ
90°
45°
-45°
10^2
10^4
10^6
ω


Parseval's Theorem
∫_{-∞}^{∞} |f(t)|² dt = 1/2π ∫_{-∞}^{∞} |F(ω)|² dω
f(t) = 1/2π ∫ F(ω) e^{jωt} dt
Proof ∫_{-∞}^{∞} |f(t)|² dt = ∫_{-∞}^{∞} f(t) (1/2π ∫_{-∞}^{∞} F(ω) e^{jωt} dω) dt
= 1/2π ∫_{-∞}^{∞} F(ω) ∫_{-∞}^{∞} f(t) e^{jωt} dt dω = 1/2π ∫_{-∞}^{∞} |F(ω)|² dω
F(-ω) = F̄(ω)
Average power P = lim_{T→∞} 1/T ∫_{-∞}^{∞} |f_T(t)|² dt
E = ∫_{-∞}^{∞} |f_T(t)|² dt = 1/2π ∫ |F(ω)|² dω
f(t)
-T/2
T/2
rect(t|T)
f_T(t)

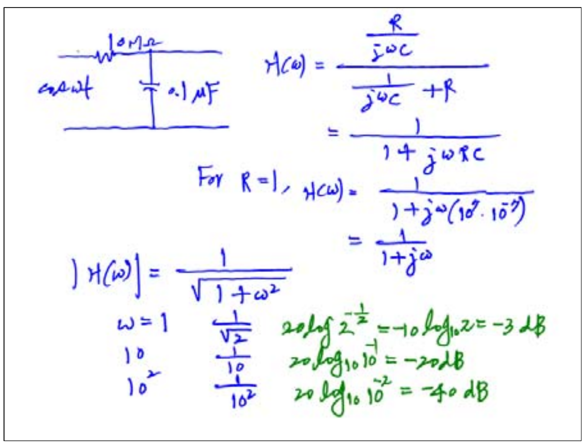

P = lim_{T→∞} 1/T ∫_{-T/2}^{T/2} |f_T(t)|² dt = lim_{T→∞} 1/2πT ∫_{-∞}^{∞} |F_T(ω)|² dω
= 1/2π ∫_{-∞}^{∞} lim_{T→∞} 1/T |F_T(ω)|² dω
P_f(ω)
P_f(ω) = lim_{T→∞} 1/T |F_T(ω)|² (Power spectral density)
Power P = 1/2π ∫_{-∞}^{∞} P_f(ω) dω
= Σ |c_k|² = 1/4π² F(0) + 1/2π² Σ_{k=1}^{∞} |F(kω0)|²
(Σ c_k e^{jkω0t} ↔ 1/2π Σ F(kω0) δ(ω - kω0))
c_k = 1/2π F(kω0)



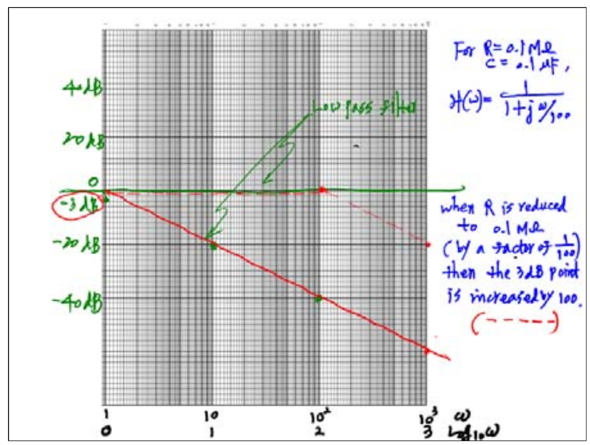
x(t) → H → y(t)
Y(ω) = H(ω) X(ω)
|Y(ω)|² = |H(ω)|² |X(ω)|²
10 log₁₀ |Y(ω)|² = 20 log₁₀ |H(ω)| + 10 log₁₀ |X(ω)|²
[dB]
20 log₁₀ |H(ω)|
40
20
-20
1
10
10²
10³
log₁₀ω



60 dB
40 dB
20 dB
-20 dB
-40 dB
1
10
10²
10³
ω
0
1
2
3
log₁₀ω



10 MΩ
cos ωt
0.1 μF
H(ω) = (1/jωC) / (1/jωC + R)
= 1 / (1 + jωRC)
For R = 1, H(ω) = 1 / (1 + jω(10⁷·10⁻⁷))
= 1 / (1+jω)
|H(ω)| = 1 / √(1+ω²)
ω = 1   1/√2   20 log 2^(-1/2) = -10 log₁₀ 2 = -3 dB
10   1/10   20 log₁₀ 10⁻¹ = -20 dB
10²   1/10²   20 log₁₀ 10⁻² = -40 dB



For R = 0.1 MΩ
C = 0.1 μF,
H(ω) = 1 / (1 + jω/100)
40 dB
20 dB
0
-3 dB
-20 dB
-40 dB
Low pass filter
when R is reduced to 0.1 MΩ (by a factor of 1/100) then the 3 dB point is increased by 100.
(- - - -)
1
10
10²
10³
ω
0
1
2
3
log₁₀ω