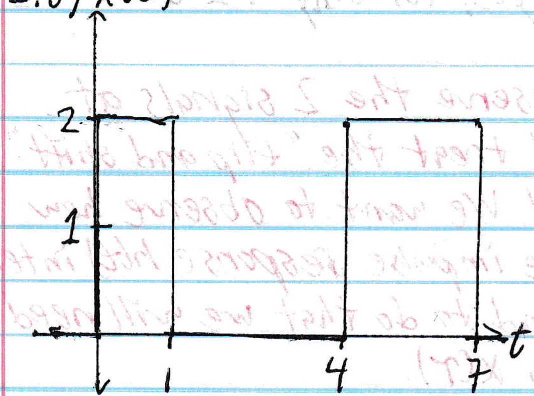
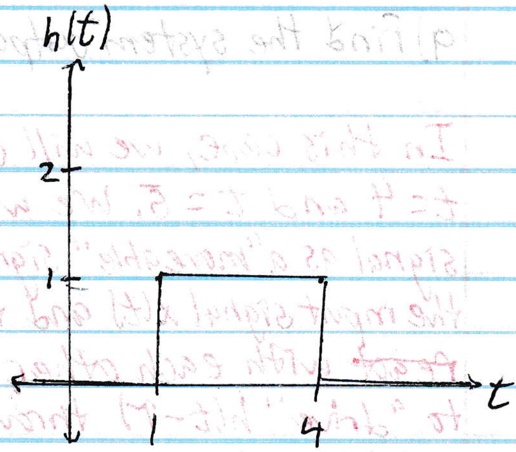
3.6)



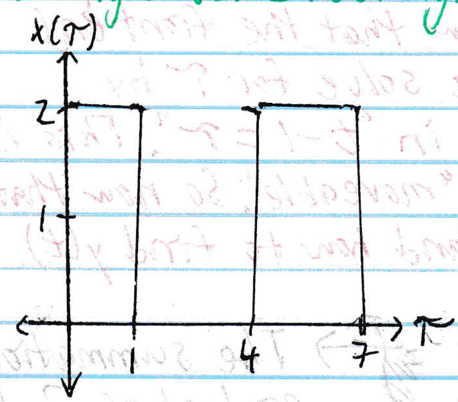


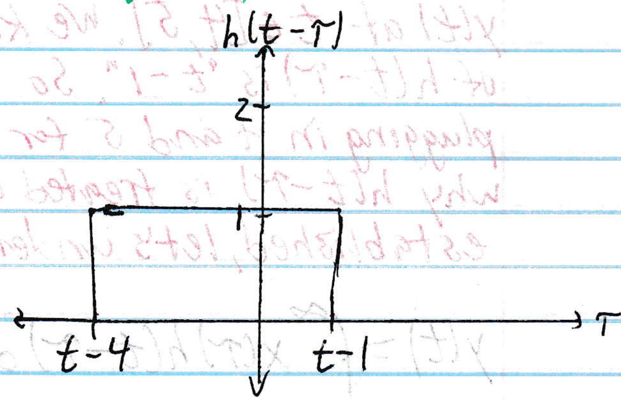
Properties of Convolution

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

What "$t-\tau$" does to the signal is that it "flips" it and "shifts" the signal. You want to apply this "flip and shift" to the signal that looks simpler. In this case, $h(t)$ is "simpler" because it's 1 rectangle vs. 2 rectangles in $x(t)$. So, let's do it!

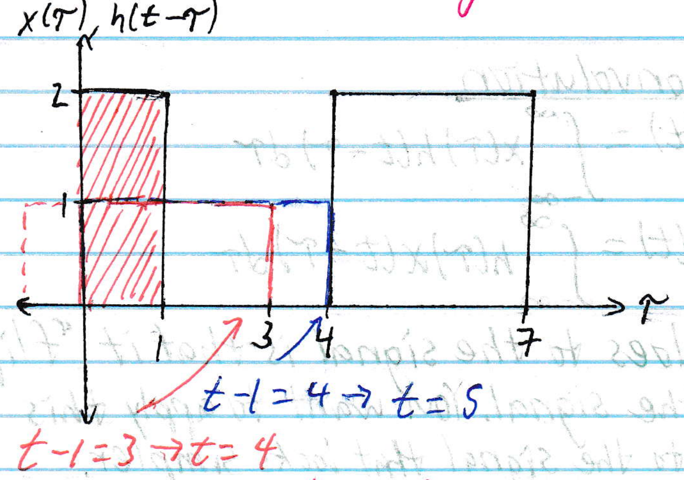




Now that we've switched to the "τ" domain, "t" is treated as a constant. What might seem weird is that $h(t-\tau)$ does not look like a "flip and shift" of $h(t)$. The current position of it is arbitrary until we actually use it. This will be explained into detail once we start the problem.

a) Find the system output $y(t)$ for only $4 \le t \le 5$

In this case, we will observe the 2 signals at $t=4$ and $t=5$. We will treat the "flip and shift" signal as a "moveable" signal. We want to observe how the input signal $x(t)$ and the impulse response $h(t)$ interact ~~react~~ with each other, and to do that we will need to "drive" $h(t-\tau)$ through $x(\tau)$.

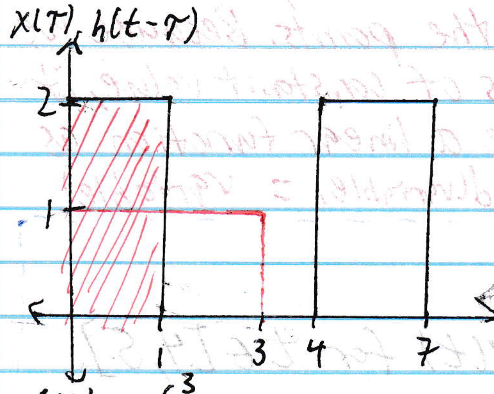


What gets a little confusing here is why $h(t-\tau)$ is at $\tau=3$ and $\tau=4$. We want to observe the output $y(t)$ at $t \in [4,5]$. We know that the front end of $h(t-\tau)$ is "$t-1$". So we solve for τ by plugging in 4 and 5 for t in "$t-1=\tau$". This is why $h(t-\tau)$ is treated as "moveable". So now that's established, let's understand how to find $y(t)$.

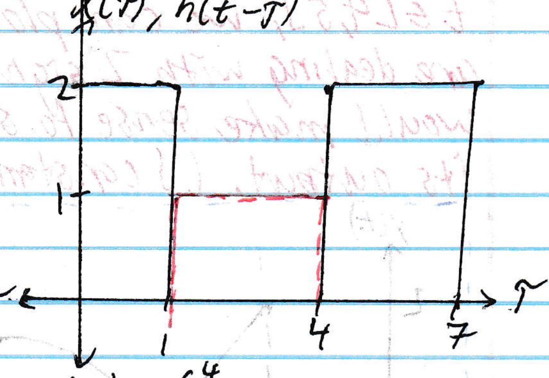
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau \rightarrow \text{The summation of the product of 2 functions}$$

In our case, we are only dealing with 2 signals with constant value. This will help in the simplicity of finding the convolution. Now I will proceed to find the values of $y(4)$ and $y(5)$ knowing this.

When $t=4$



When $t=5$



$$y(4)=\int_0^3 x(\tau)h(t-\tau)\,d\tau$$

$$y(4)=\int_0^1 (2)(1)\,d\tau$$

$$y(4)=2\int_0^1 d\tau = 2$$

$$y(5)=\int_1^4 x(\tau)h(t-\tau)\,d\tau$$

$$y(5)=\int_1^4 (0)(1)\,d\tau$$

$$y(5)=0$$

I sort of jumped the gun with the integral limits. The generalized form of the integral is the following:

$$y(t)=\int_{t-4}^{t-1} x(\tau)h(t-\tau)\,d\tau$$

The reasoning is that our moveable signal is bounded by its dimensions. These dimensions permanently becomes its integral bounds for convolution. I will demonstrate the process accordingly, to see how I created my previous integrals. I should have done this first.

$$y(t)=\int_{t-4}^{t-1} x(\tau)h(t-\tau)\,d\tau$$ Generalized Convolution Integral for $h(t-\tau)$

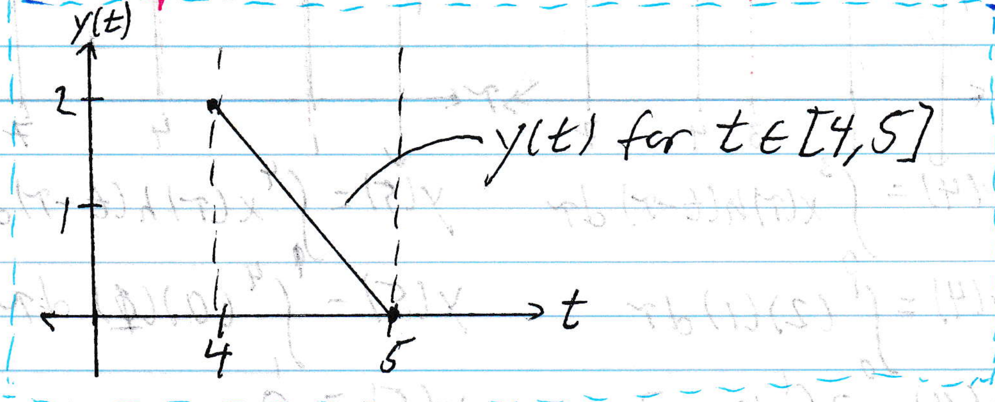
$$y(4)=\int_0^3 x(\tau)h(t-\tau)\,d\tau$$ Input the values for "t"

$$y(4)=\int_0^1 (2)(1)\,d\tau+\int_1^3 (0)(1)\,d\tau$$ Replace functions with values (IF Constant)

↳ Break Up Integral Properly

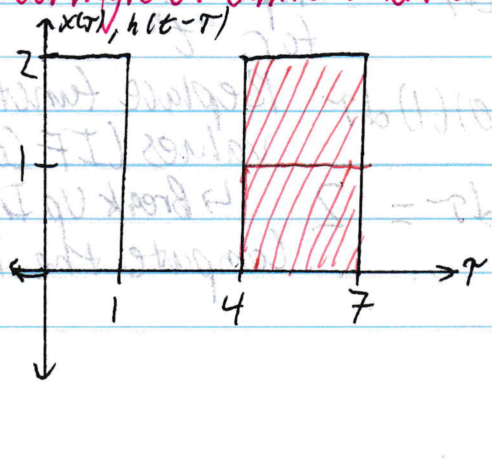
$$y(4)=\int_0^1 2\,d\tau = 2\int_0^1 d\tau = 2$$ Compute the Integral

Now that we have the values for $y(t)$ between $t \in [4,5]$, we can plot the points. Because we are dealing with 2 signals of constant value, it would make sense to see a linear function as its output. ($\int$ constant d(variable) = variable)



b) Find the maximum value of the output $y(t)$

We observed that convolution is just the Area under the product of 2 signals. To obtain its maximum value for the output, it is when our "moveable" signal is underneath the maximum values of our "static" signal. If this seems confusing, I'm sorry, but it'll make more sense once we meet in person! I can explain it better in person than on paper, I think. In this case, $h(t-\tau)$ is a rectangle of width "3". To achieve the maximum value, it will fit itself between the rectangle of same width on ~~x(t)~~ $x(\tau)$.



$$Y_{max}(t) = \int_4^7 (2)(1)\, d\tau$$

$$Y_{max}(t) = 2\int_4^7 d\tau = 2(7-4)$$

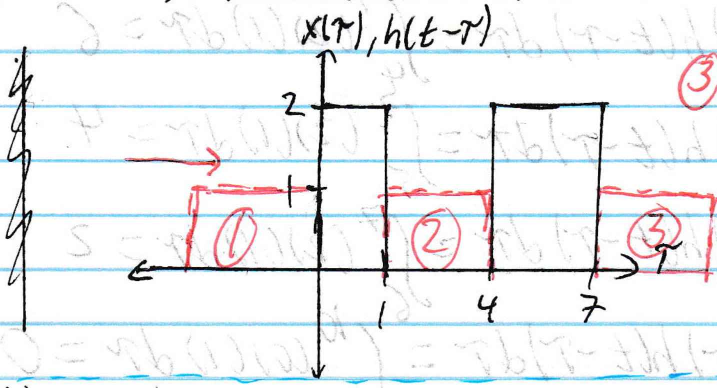
$$\boxed{Y_{max}(t) = 6}$$

c) Find the ranges of time for which the output is zero.

The only time the output is zero is when $h(t-\tau)$ and its entirety is interacting with $x(t)=0$. That means that not even a sliver of $h(t-\tau)$ is underneath/crossing a value where $x(t) \neq 0$. There are 3 occurences where this happens.

① $\tau \in (-\infty, 0) \rightarrow t \in (-\infty, 1)$ ② $\tau = 4 \rightarrow t = 5$

③ $\tau \in (10, \infty) \rightarrow t \in (11, \infty)$



The output of our system is 0 when $t \in (-\infty, 1)$, $t=5$, and $t \in (11, \infty)$.

d) Solve and sketch $y(t)$ for all time to verify all results.

Now we will use the generalized form to plot multiple points for our system output.

$$y(t) = \int_{t-4}^{t-1} x(\tau)h(t-\tau)\,d\tau$$

$$y(1) = \int_{-3}^{0} x(\tau)h(t-\tau)\,d\tau = \int_{-3}^{0} (0)(1)\,d\tau = 0$$

$$y(2) = \int_{-2}^{1} x(\tau)h(t-\tau)\,d\tau = \int_{0}^{1} (2)(1)\,d\tau = 2$$

$$y(3) = \int_{-1}^{2} x(\tau)h(t-\tau)\,d\tau = \int_{0}^{1} (2)(1)\,d\tau = 2$$

$$y(4) = \int_0^3 x(\tau)h(t-\tau)\,d\tau = \int_0^1 (2)(1)\,d\tau = 2$$

$$y(5) = \int_1^4 x(\tau)h(t-\tau)\,d\tau = \int_1^4 (0)(1)\,d\tau = 0$$

$$y(6) = \int_2^5 x(\tau)h(t-\tau)\,d\tau = \int_4^5 (2)(1)\,d\tau = 2$$

$$y(7) = \int_3^6 x(\tau)h(t-\tau)\,d\tau = \int_4^6 (2)(1)\,d\tau = 4$$

$$y(8) = \int_4^7 x(\tau)h(t-\tau)\,d\tau = \int_4^7 (2)(1)\,d\tau = 6$$

$$y(9) = \int_5^8 x(\tau)h(t-\tau)\,d\tau = \int_5^7 (2)(1)\,d\tau = 4$$

$$y(10) = \int_6^9 x(\tau)h(t-\tau)\,d\tau = \int_6^7 (2)(1)\,d\tau = 2$$

$$y(11) = \int_7^{10} x(\tau)h(t-\tau)\,d\tau = \int_7^{10} (0)(1)\,d\tau = 0$$

Now that we have the values, let's plot y(t)!